

PHASE TRANSITION OF A HEAT EQUATION WITH ROBIN'S BOUNDARY CONDITIONS AND EXCLUSION PROCESS

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ABSTRACT. We consider the exclusion process evolving in the one-dimensional discrete torus, with a bond whose conductance slows down the passage of particles across it. We chose the conductance at that bond as $\alpha n^{-\beta}$, where $\alpha > 0$, $\beta \in [0, \infty]$, and n is the scale parameter. In [3], by rescaling time diffusively, it was proved that the hydrodynamical limit depends strongly on the regime of β . Here, firstly we derive a new proof of the hydrodynamical limit for $\beta = 1$, by showing that the hydrodynamic equation, is a Heat Equation with Robin's boundary conditions that depend on α . As a consequence, the weak solution of the hydrodynamic equation given in [3], involving a generalized derivative $\frac{d}{du} \frac{d}{dW}$, coincides with the weak solution of a Heat Equation with Robin's boundary conditions. Secondly, arguing by energy estimates, we prove a phase transition for the weak solution of a Heat Equation with Robin's boundary conditions. Namely, if $\alpha \rightarrow \infty$, that weak solution converges to the weak solution of the Heat Equation without boundary conditions, while if $\alpha \rightarrow 0$, the convergence is to the weak solution of the Heat Equation with Neumann's boundary conditions.

1. INTRODUCTION

The characterization of scaling limits of discrete systems is a central question in Statistical Mechanics. In the case of Interacting Particle Systems, where particles evolve according to some rule of interaction, it is of interest to characterize, in the continuum limit, the time trajectory of the spatial density of particles, the so called *hydrodynamical limit*. As a reference on the subject, see [6].

Among the several kinds of interacting particle systems, the *Exclusion Process* is of special importance in Statistical Mechanics and Probability. The Exclusion Process is a Markov process where each particle waits a mean one exponential time, after which moves as a random walk, but the jump is performed if and only if the destination site is empty, otherwise the particle has to wait a new random time. This is the *exclusion rule* that particles have to obey. Jumps can occur from a site x to a site y with a jump probability $p(x, y)$ that has to turn the process well defined, see [5]. When jumps occur to nearest-neighbor sites, the process is known by *Simple Exclusion*. In Physics, particles obeying such exclusion rule, are called *fermions*.

In [3], it was considered the Exclusion Process in presence of a finite number of slow bonds. In this process all bonds have conductance equal to 1, except the *slow bond* whose conductance is given by $\alpha n^{-\beta}$, where $\alpha > 0$, $\beta \in [0, \infty]$ and n is the scale parameter. Its dynamics can be described as follows. In the one-dimensional

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discrete torus with n sites, that we denote by \mathbb{T}_n , it is allowed at most one particle per site. Associated to each bond of \mathbb{T}_n , there exists a Poisson process of time arrivals, all of them independent. At the time arrival of any Poisson process, the occupation at the vertices of the corresponding bond are interchanged. The Poisson processes have all parameter 1, except a particular Poisson process. This particular Poisson process is assumed to have parameter $\alpha n^{-\beta}$, where $\alpha > 0$ and $\beta \in [0, \infty]$. Since for n big $\alpha n^{-\beta}$ is much smaller than 1, this bond slows down the passage of particles across it, and that is the reason for its name *slow bond*.

In [3], it was proved that the hydrodynamical limit for that process, exhibits three different macroscopic behaviors depending on the regime of β . If $\beta \in [0, 1)$, the limit time trajectory of the spatial density of particles is given by the Heat Equation:

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), \\ \rho(0, u) = \rho_0(u), \end{cases} \quad (1)$$

$\forall t \in [0, T]$ and $\forall u \in \mathbb{T}$, where $\Delta = \partial_u^2$ and $\mathbb{T} = [0, 1)$ is the one-dimensional torus in which 0 is identified with 1. If $\beta = 1$, the limit time trajectory of the spatial density of particles is given by:

$$\begin{cases} \partial_t \rho(t, u) = \frac{d}{du} \frac{d}{dW} \rho(t, u), \\ \rho(0, u) = \rho_0(u), \end{cases} \quad (2)$$

$\forall t \in [0, T]$ and $\forall u \in \mathbb{T}$, where $\frac{d}{du} \frac{d}{dW}$ is a generalized derivative, being W a measure given by the sum of the Lebesgue measure and a delta of Dirac. For $\beta \in (1, \infty)$, the limit time trajectory of the spatial density of particles is given by the Heat Equation with Neumann's boundary conditions:

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), \\ \rho(0, u) = \rho_0(u), \\ \partial_u \rho(t, b^+) = \partial_u \rho(t, b^-) = 0, \end{cases} \quad (3)$$

$\forall t \in [0, T]$ and $\forall u \in \mathbb{T}$, where $b \in \mathbb{T}$. We consider the exclusion process of [3] with a unique slow bond $\{b_n, b_n + 1\}$ that corresponds to the macroscopic point $b \in \mathbb{T}$. Here and in the sequel the notation b^- or b^+ means that we are taking spacial lateral limits, from the left or right of b , respectively, see (9). These three different macroscopical behaviors correspond to a dynamical phase transition coming from the rule governing the system at the microscopical level.

The fact of having a system exhibiting that dynamical phase transition, namely starting from the classical Heat Equation (1), arriving in equation (2) involving a generalized derivative $\frac{d}{du} \frac{d}{dW}$, and then obtaining again the Heat Equation but with Neumann's boundary conditions, puzzled us in the sense that we expected that at the critical level $\beta = 1$, one should obtain in the hydrodynamics, the Heat Equation with some boundary conditions that would interpolate from Neumann's boundary conditions to no conditions at all.

Also, inspired by our work [3], another question was naturally raised: is there a corresponding dynamical phase transition at the macroscopical level? Would the solutions of (1), (2), (3) be continuously related to some parameter given at the boundary conditions? Notice that these questions are concerned only with the partial differential equations, having, at principle, no relation at all with the underlying particle system.

The main result of this paper is a positive answer to all the questions raised above. First, considering a slow bond $\{b_n, b_n + 1\}$ corresponding to the macroscopic point $b \in \mathbb{T}$ of intensity $\alpha n^{-\beta}$, where $\alpha > 0$, we present another proof for the hydrodynamical limit in the regime $\beta = 1$, arriving at the Heat Equation with Robin's boundary conditions given by

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), \\ \rho(0, u) = \rho_0(u), \\ \partial_u \rho(t, b^+) = \partial_u \rho(t, b^-) = \alpha(\rho(t, b^+) - \rho(t, b^-)), \end{cases} \quad (4)$$

$\forall t \in [0, T]$ and $\forall u \in \mathbb{T}$, where $b \in \mathbb{T}$.

An immediate and important consequence of last result, is that the weak solution of the generalized equation (2) coincides with the weak solution of the classical Heat Equation (4) with $\alpha = 1$. Moreover, an interesting phenomena which follows from the previous result is that the only case in which one gets the dependence on α in the macroscopic equation, is for $\beta = 1$. In the remaining cases the presence of α is completely negligible. We notice that for $\beta \neq 1$, the hydrodynamic behavior is as given in [3].

Next, we index the weak solution of (4) in α , denoting it by ρ^α . We also denote by ρ^0 the solution of (1) and by ρ^∞ the solution of (3). Based on energy estimates coming from the underlying particle system and the hydrodynamical limit, we were able to prove the following convergence in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$:

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \rho^\alpha &= \rho^0, \\ \lim_{\alpha \rightarrow \infty} \rho^\alpha &= \rho^\infty. \end{aligned} \quad (5)$$

To our knowledge, it is presented here for the first time, the derivation of a phase transition for the weak solution of the Heat Equation with Robin's boundary conditions, using the exclusion process as an approximating stochastic model and the framework of probability theory to obtain knowledge on the behavior of this solution.

Here follows an outline of this work. Precise definitions, statements and results are given in Section 2. In Section 3, we establish the relation between equations (2) and (4) and we also prove the hydrodynamic limit for the exclusion process with a slow bond of intensity $\alpha n^{-\beta}$. There, it is also mentioned the hydrodynamic limit for $\beta = \infty$, not considered in [3], which corresponds to the exclusion process evolving in a discrete box with isolated boundaries. Section 4 is devoted to the proof of uniqueness of weak solutions of (4) and we also present a simple proof of uniqueness of weak solutions of (1) and (3). Section 5 and Section 6 are devoted to the proof of the phase transition for the Heat Equation with Robin's boundary conditions, or else, the convergence (5). Finally, in the Appendix, we present some results about Sobolev spaces needed in due course.

2. STATEMENT OF RESULTS

2.1. Slowed exclusion process.

The exclusion process evolving on $\mathbb{T}_n = \mathbb{Z}/n\mathbb{Z}$, the one-dimensional discrete torus with n points, is described as follows. It is the Markov process $\{\eta_t : t \geq 0\}$

with state space $\{0, 1\}^{\mathbb{T}_n}$ and generator \mathcal{L}_n acting on functions $f : \{0, 1\}^{\mathbb{T}_n} \rightarrow \mathbb{R}$ as

$$\mathcal{L}_n f(\eta) = \sum_{x \in \mathbb{T}_n} \xi_{x,x+1}^n [f(\eta^{x,x+1}) - f(\eta)],$$

where $\xi_{x,x+1}^n = \xi_{x+1,x}^n \geq 0$ is the conductance at the bond $\{x, x+1\}$ and $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+1)$, namely

$$\eta^{x,x+1}(y) = \begin{cases} \eta(x+1), & \text{if } y = x, \\ \eta(x), & \text{if } y = x+1, \\ \eta(y), & \text{otherwise.} \end{cases}$$

Its dynamics can be informally described as follows. At each bond $\{x, x+1\}$, there is an exponential clock of parameter $\xi_{x,x+1}^n$. When this clock rings, the value of η at the vertices of this bond are exchanged. This means that particles can cross the bond $\{x, x+1\}$ at rate $\xi_{x,x+1}^n$. Throughout the paper $\xi_{x,x+1}^n > 0$, nevertheless if $\xi_{x,x+1}^n = 0$, then the passage of particles across the bond $\{x, x+1\}$ is forbidden. One can interpret that bond as a barrier blocking the passage of particles from one region to another.

Now, we specify the conductances that suit our purposes. Fix $\alpha > 0$, $\beta \in [0, \infty]$ and $x \in \mathbb{T}_n$. The conductances, $\xi_{x,x+1}^n = \xi_{x,x+1}^{n,\beta,\alpha}$, of the exclusion process with a single slow bond $\{b_n, b_n+1\}$ that corresponds to the macroscopic point $b \in \mathbb{T}$, where \mathbb{T} denotes the one-dimensional continuous torus $[0, 1)$, are given by:

$$\xi_{x,x+1}^n = \begin{cases} \alpha n^{-\beta}, & \text{if } x = b_n, \\ 1, & \text{otherwise.} \end{cases}$$

The dynamics of the exclusion process with these conductances is such that particles cross all the bonds at rate one, except the bond $\{b_n, b_n+1\}$ whose dynamics is slowed down as $\alpha n^{-\beta}$, with $\alpha > 0$ and $\beta \in [0, \infty]$. It is understood here that $n^{-\infty} = 0$ and $\infty \cdot 0 = 0$.

It is well known that the Bernoulli product measures on $\{0, 1\}^{\mathbb{T}_n}$ with constant parameter $\gamma \in [0, 1]$, denoted by $\{\nu_\gamma^n : 0 \leq \gamma \leq 1\}$, are invariant for the dynamic introduced above. Moreover, since $\xi_{x,x+1}^n = \xi_{x+1,x}^n$ for all $x \in \mathbb{T}_n$ these measures are also reversible.

In order to keep notation simple, throughout the text we write $\eta_t := \eta_{tn^2}$. Then, $\{\eta_t : t \geq 0\}$ turns out to be the Markov process on $\{0, 1\}^{\mathbb{T}_n}$ associated to the generator \mathcal{L}_n speeded up by n^2 . Also we do not index the process neither in β nor in α .

The trajectories of our Markov process live on the space $\mathcal{D}(\mathbb{R}_+, \{0, 1\}^{\mathbb{T}_n})$, i.e., the path space of càdlàg trajectories with values in $\{0, 1\}^{\mathbb{T}_n}$. For a measure μ_n on $\{0, 1\}^{\mathbb{T}_n}$, we denote by $\mathbb{P}_{\mu_n}^{\alpha,\beta}$ the probability measure on $\mathcal{D}(\mathbb{R}_+, \{0, 1\}^{\mathbb{T}_n})$ induced by the initial distribution μ_n and the Markov process $\{\eta_t : t \geq 0\}$ and we denote by $\mathbb{E}_{\mu_n}^{\alpha,\beta}$ expectation with respect to $\mathbb{P}_{\mu_n}^{\alpha,\beta}$.

In order to state our first result we need to impose some conditions on the initial distribution of the system.

Definition 1. A sequence of probability measures $\{\mu_n : n \geq 1\}$ on $\{0, 1\}^{\mathbb{T}_n}$ is said to be associated to a profile $\rho_0 : \mathbb{T} \rightarrow [0, 1]$ if, for every $\delta > 0$ and every $H \in C(\mathbb{T})$,

$$\lim_{n \rightarrow +\infty} \mu_n \left\{ \eta : \left| \frac{1}{n} \sum_{x \in \mathbb{T}_n} H\left(\frac{x}{n}\right) \eta(x) - \int_{\mathbb{T}} H(u) \rho_0(u) du \right| > \delta \right\} = 0, \quad (6)$$

where for $n \in \mathbb{N}_0$, $C^n(\mathbb{T})$ denotes the set of continuous functions from \mathbb{T} to \mathbb{R} and with continuous derivatives of order up to n and $C^0(\mathbb{T})$ is simply denoted by $C(\mathbb{T})$.

2.2. The hydrodynamical equations.

In this section we present the partial differential equations that govern the evolution of the limiting density profile. These are the *hydrodynamic equations* of the slowed exclusion process introduced above.

Definition 2. (Heat Equation)

Fix $\rho_0 : \mathbb{T} \rightarrow [0, 1]$. The Heat Equation with initial condition $\rho_0(\cdot)$ is given by

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), \\ \rho(0, u) = \rho_0(u), \end{cases} \quad (7)$$

$\forall t \in [0, T]$ and $\forall u \in \mathbb{T}$, where $\Delta = \partial_u^2$.

Definition 3. (Heat Equation with Robin's boundary conditions)

Fix $b \in \mathbb{T}$ and $\rho_0 : \mathbb{T} \rightarrow [0, 1]$. We consider a Heat Equation with Robin's boundary conditions at $b \in \mathbb{T}$ and with initial condition $\rho_0(\cdot)$ given by

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), \\ \rho(0, u) = \rho_0(u), \\ \partial_u \rho(t, b^+) = \partial_u \rho(t, b^-) = \alpha(\rho(t, b^+) - \rho(t, b^-)), \end{cases} \quad (8)$$

$\forall t \in [0, T]$ and $\forall u \in \mathbb{T}$.

As mentioned in the introduction, here and in the sequel, whenever we use the notation b^- or b^+ we mean that we are taking the spacial lateral limits, from the left or right of b , respectively. So that,

$$\begin{aligned} \partial_u \rho(t, b^+) &= \lim_{\substack{u \rightarrow b \\ u > b}} \partial_u \rho(t, u), & \partial_u \rho(t, b^-) &= \lim_{\substack{u \rightarrow b \\ u < b}} \partial_u \rho(t, u), \\ \rho(t, b^+) &= \lim_{\substack{u \rightarrow b \\ u > b}} \rho(t, u), & \rho(t, b^-) &= \lim_{\substack{u \rightarrow b \\ u < b}} \rho(t, u). \end{aligned} \quad (9)$$

Definition 4. (Heat Equation with Neumann's boundary conditions)

Fix $b \in \mathbb{T}$ and $\rho_0 : \mathbb{T} \rightarrow [0, 1]$. The Heat Equation with Neumann's boundary conditions at $b \in \mathbb{T}$ and with initial condition $\rho_0(\cdot)$ is given by

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u), \\ \rho(0, u) = \rho_0(u), \\ \partial_u \rho(t, b^+) = \partial_u \rho(t, b^-) = 0, \end{cases} \quad (10)$$

$\forall t \in [0, T]$ and $\forall u \in \mathbb{T}$.

2.3. Weak solutions.

Now we give the precise definition of weak solutions to these equations. We start by the classical Heat Equation. For \mathcal{I} an interval of \mathbb{T} , here and in the sequel for $n, m \in \mathbb{N}$, $C^{n,m}([0, T] \times \mathcal{I})$ denotes the set of functions defined on $[0, T] \times \mathcal{I}$, of class C^n in time and C^m in space.

Let μ be a measure on \mathbb{T} . For $H, G : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ and $s \in [0, T]$, we use the notation

$$\langle H_s, G_s \rangle_\mu = \int_{\mathbb{T}} H_s(u) G_s(u) \mu(du).$$

If μ is Lebesgue, we omit the subindex μ in the definition above and we use the notation $\|H\|_{L^2(\mathbb{T})} := \langle H, H \rangle$. Here and in the sequel, a subindex in a function means a variable, *not a derivative*. For instance, above in H_s , we mean that $H_s(u) = H(s, u)$.

Definition 5. (*Weak solution of Heat Equation*)

A bounded function $\rho : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ is a weak solution of (7) if, for any $t \in [0, T]$ and any $H \in C^{1,2}([0, T] \times \mathbb{T})$, $\rho(t, \cdot)$ satisfies the integral equation

$$\langle \rho_t, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s, \partial_s H_s + \Delta H_s \rangle ds = 0. \quad (11)$$

In order to state the notion of weak solutions of the Heat Equation with Robin's or Neumann's boundary conditions we need to introduce a suitable space for its solutions.

Definition 6. (*Sobolev space*)

Fix $a, b \in \mathbb{T}$, with $a < b$. The Sobolev space $\mathcal{H}^1(a, b)$ consists of all locally summable functions $\zeta : (a, b) \rightarrow \mathbb{R}$ such that there exists $\partial \zeta \in L^2(a, b)$ satisfying

$$\langle \partial_u G, \zeta \rangle = -\langle G, \partial \zeta \rangle,$$

for all $G \in C^\infty(a, b)$ with compact support. For $\zeta \in \mathcal{H}^1(a, b)$, we define the norm

$$\|\zeta\|_{\mathcal{H}^1(a, b)} = \|\partial \zeta\|_{L^2(a, b)}.$$

Definition 7. Fix $a, b \in \mathbb{T}$, with $a < b$. The space $L^2(0, T; \mathcal{H}^1(a, b))$ consists of all measurable functions $\xi : [0, T] \rightarrow \mathcal{H}^1(a, b)$ with

$$\|\xi\|_{L^2(0, T; \mathcal{H}^1(a, b))}^2 := \int_0^T \|\xi_t\|_{\mathcal{H}^1(a, b)}^2 dt < \infty.$$

The study of the slowed exclusion process requires the use of test functions defined on $[0, T] \times \mathbb{T}$, which must be smooth except possibly at $b \in \mathbb{T}$. For that purpose, we introduce the following space.

Definition 8. Denote by $C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$ the space of functions $H : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ that satisfy

- $H \in C^{1,2}([0, T] \times (b, 1 + b))$;
- there exists a function $\tilde{H} : [0, T] \times [b, 1 + b] \rightarrow \mathbb{R}$ such that:
 1. $\tilde{H} \in C^{1,2}([0, T] \times [b, 1 + b])$;
 2. \tilde{H} restricted to $[0, T] \times (b, 1 + b)$ coincides with H .

Here we are identifying $(b, 1 + b)$ with $\mathbb{T} \setminus \{b\}$.

Notice that $C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$ should not be misunderstood with $C^{1,2}([0, T] \times \mathbb{T})$, since a typical function of $C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$ may have a discontinuity at $b \in \mathbb{T}$.

Definition 9. (*Weak solution of Heat Equation with Robin's boundary conditions*)

Fix $b \in \mathbb{T}$ and $\alpha > 0$. A bounded function $\rho : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ is a weak solution of (8), if the following two conditions are fulfilled:

- (1) $\rho \in L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$;

(2) For any $t \in [0, T]$ and any $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$, $\rho(t, \cdot)$ satisfies the integral equation

$$\begin{aligned} & \langle \rho_t, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s, \partial_s H_s + \Delta H_s \rangle ds \\ & - \int_0^t \{ \rho_s(b^+) \partial_u H_s(b^+) - \rho_s(b^-) \partial_u H_s(b^-) \} ds \\ & + \int_0^t \alpha \{ \rho_s(b^+) - \rho_s(b^-) \} \{ H_s(b^+) - H_s(b^-) \} ds = 0. \end{aligned} \quad (12)$$

We notice that the space $\mathcal{H}^1(\mathbb{T} \setminus \{b\})$ is in fact the space $\mathcal{H}^1(b, 1+b)$.

Definition 10. (*Weak solution of Heat Equation with Neumann's boundary conditions*)

Fix $b \in \mathbb{T}$. A bounded function $\rho : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ is a weak solution of (10), if the following two conditions are fulfilled:

(1) $\rho \in L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$;

(2) For any $t \in [0, T]$ and any $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$, $\rho(t, \cdot)$ satisfies the integral equation

$$\begin{aligned} & \langle \rho_t, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s, \partial_s H_s + \Delta H_s \rangle ds \\ & - \int_0^t \{ \rho_s(b^+) \partial_u H_s(b^+) - \rho_s(b^-) \partial_u H_s(b^-) \} ds = 0. \end{aligned} \quad (13)$$

We notice that last equation coincides with (12) for $\alpha = 0$. For classical results about Sobolev spaces, we refer to [1] and [4]. Since in Definitions 9 and 10 we imposed $\rho \in L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$, the integrals above are well-defined at boundary points. Heuristically, in order to establish an integral equation for the weak solution of the Heat Equation with Robin's (resp. Neumann's) boundary conditions as above, one should multiply both sides of the first equation in (8) (resp. (10)) by a test function H , then integrate both in space and time and finally, perform twice a formal integration by parts. Then, applying the formal Robin's (resp. Neumann's) boundary conditions to ρ , one gets to (12) (resp. (13)). Moreover, any strong solution of (8) (resp. (10)) is a weak solution of (8) (resp. (10)).

2.4. Hydrodynamical phase transition.

In order to establish the hydrodynamical limit we introduce the empirical measure process as follows. We denote by \mathcal{M} the space of positive measures on \mathbb{T} with total mass bounded by one, endowed with the weak topology. For $\eta \in \{0, 1\}^{\mathbb{T}_n}$, let $\pi^n(\eta, \cdot) \in \mathcal{M}$ be the empirical measure, namely, the measure on \mathbb{T} obtained by rescaling space by n and by assigning mass n^{-1} to each particle of η :

$$\pi^n(\eta, du) = \frac{1}{n} \sum_{x \in \mathbb{T}_n} \eta(x) \delta_{x/n}(du),$$

where δ_y means the Dirac measure concentrated on $y \in \mathbb{T}$. For $t \in [0, T]$, let $\pi_t^n(\eta, du) := \pi^n(\eta_t, du)$. For a test function $H : \mathbb{T} \rightarrow \mathbb{R}$ we use the following notation

$$\langle \pi_t^n, H \rangle := \int H(u) \pi_t^n(\eta, du) = \frac{1}{n} \sum_{x \in \mathbb{T}_n} H\left(\frac{x}{n}\right) \eta_t(x).$$

We use this notation since for π_t absolutely continuous with respect to the Lebesgue measure with density ρ_t , we write $\langle \rho_t, H \rangle$ for $\langle \pi_t, H \rangle$.

Fix $T > 0$. Let $\mathcal{D}([0, T], \mathcal{M})$ be the space of càdlàg trajectories with values in \mathcal{M} and endowed with the *Skorohod* topology. For each probability measure μ_n on $\{0, 1\}^{\mathbb{T}_n}$, denote by $\mathbb{Q}_{n, \mu_n}^{\alpha, \beta}$ the measure on the path space $\mathcal{D}([0, T], \mathcal{M})$ induced by μ_n and the empirical process π_t^n introduced above.

Now, we state the dynamical phase transition at the hydrodynamics level for the slowed exclusion process introduced above. We notice that this result is an improvement of the main theorem of [3].

Theorem 2.1. *Fix $\beta \in [0, \infty]$. Consider the exclusion process with a single slow bond corresponding to the macroscopic point $b \in \mathbb{T}$ and with conductance $\alpha n^{-\beta}$ at the corresponding microscopic bond $\{b_n, b_n + 1\}$, with $\alpha > 0$. Fix a continuous profile $\rho_0 : \mathbb{T} \rightarrow [0, 1]$ and let $\{\mu_n : n \geq 1\}$ be a sequence of probability measures on $\{0, 1\}^{\mathbb{T}_n}$ associated to $\rho_0(\cdot)$. Then, for any $t \in [0, T]$, for every $\delta > 0$ and every $H \in C(\mathbb{T})$:*

$$\lim_{n \rightarrow +\infty} \mathbb{P}_{\mu_n}^{\alpha, \beta} \left\{ \eta : \left| \frac{1}{n} \sum_{x \in \mathbb{T}_n} H\left(\frac{x}{n}\right) \eta_t(x) - \int_{\mathbb{T}} H(u) \rho(t, u) du \right| > \delta \right\} = 0,$$

where:

- if $\beta \in [0, 1)$, $\rho(t, \cdot)$ is the unique weak solution of (7);
- if $\beta = 1$, $\rho(t, \cdot)$ is the unique weak solution of (8);
- if $\beta \in (1, \infty]$, $\rho(t, \cdot)$ is the unique weak solution of (10).

The proof of last result is given in [3] for $\beta \in [0, 1)$ and $\beta \in (1, \infty)$. For $\beta = 1$, the proof in [3] can be almost all adapted to our case here, except the characterization of limit points. Here we characterize the limit points by identifying them as weak solutions of the Heat Equation with Robin's boundary conditions given in (8). The proof of this result is proved in Section 3.

For $\beta = \infty$, the same arguments as used in [3] for $\beta \in (1, \infty)$ fit the case $\beta = \infty$ and for that reason we also omit the proof in this case.

2.5. Phase transition for the Heat Equation with Robin's boundary conditions.

Let μ be a measure on \mathbb{T} . For $H, G : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$, we use the notation

$$\langle\langle H, G \rangle\rangle_{\mu} = \int_0^T \langle H_s, G_s \rangle_{\mu} ds = \int_0^T \int_{\mathbb{T}} H_s(u) G_s(u) \mu(du) ds.$$

If μ is the Lebesgue measure, we omit the subindex μ in the definition above.

Define the measure

$$W_{\alpha}(du) = du + \frac{1}{\alpha} \delta_b(du), \quad (14)$$

that is, W_{α} is the sum of the Lebesgue measure and the Dirac measure concentrated on b with weight $1/\alpha$. We denote by $L_{W_{\alpha}}^2([0, T] \times \mathbb{T})$ the Hilbert space composed of measurable functions $f : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ with $\|f\|_{W_{\alpha}}^2 := \langle\langle f, f \rangle\rangle_{W_{\alpha}} < \infty$. Observe that functions in $L_{W_{\alpha}}^2([0, T] \times \mathbb{T})$ have their value at b uniquely defined. When W is the Lebesgue measure we simply denote by $L^2([0, T] \times \mathbb{T})$ the Hilbert space composed of measurable functions $f : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ with $\|f\|^2 := \langle\langle f, f \rangle\rangle < \infty$.

Theorem 2.2. *For any $\alpha > 0$, there exists a weak solution $\rho^\alpha : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ of (8). Moreover, such solution is unique and satisfies the inequality*

$$\sup_{H \in C^{0,1}([0, T] \times \mathbb{T})} \left\{ \langle \rho^\alpha, \partial_u H \rangle - 2 \langle H, H \rangle_{W_\alpha} \right\} \leq K_0,$$

where K_0 is a constant that does not depend on α .

Theorem 2.3. *For $\alpha > 0$, let $\rho^\alpha : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ be the unique weak solution of (8). Then,*

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \rho^\alpha &= \rho^0, \\ \lim_{\alpha \rightarrow \infty} \rho^\alpha &= \rho^\infty, \end{aligned}$$

in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$, where ρ^0 and ρ^∞ are the unique weak solutions of equations (10) and (7), respectively.

The proof of Theorem 2.2 is given in Section 5 and the proof of Theorem 2.3 is given in Section 6. We point out that Theorem 2.2 is a consequence of the energy estimates obtained by means of the slowed exclusion process, and it is the key in the proof of Theorem 2.3.

3. HYDRODYNAMIC LIMIT FOR $\beta = 1$

The method of proof of the hydrodynamic limit followed here is the usual in stochastic process: tightness, which means relative compactness, plus uniqueness of limit points. We recall that the proof of tightness for this case is very similar to the one given in [3] and for that reason we omitted it. So it remains to show uniqueness of limit points in order to conclude. Before proceeding with the characterization of limit points we make some remarks relatively to what we did in [3].

3.1. Relating equations (2) and (4).

In this section we are going to described how to get the weak solution of (2) from the weak solution of equation (4). In [3], for $\beta = 1$, the set of test functions for the corresponding hydrodynamic equation is \mathcal{H}_W^1 . Since in this case we need to deal with the measure W_α , we adapt the definition of \mathcal{H}_W^1 from [3] to this setting:

Definition 11. *Let $\mathcal{H}_{W_\alpha}^1$ be the set of functions H in $L^2(\mathbb{T})$ such that for $u \in \mathbb{T}$*

$$H(u) = \tilde{a} + \int_{(0, u]} \left(\tilde{b} + \int_0^v h(w) dw \right) W_\alpha(dv), \quad (15)$$

for some function h in $L^2(\mathbb{T})$ and $\tilde{a}, \tilde{b} \in \mathbb{R}$ such that

$$\int_0^1 h(u) du = 0, \quad \int_{(0, 1]} \left(\tilde{b} + \int_0^v h(w) dw \right) W_\alpha(dv) = 0, \quad (16)$$

where W_α was given in (14). We define \mathcal{C}_{W_α} as the set of functions $H \in \mathcal{H}_{W_\alpha}^1$ such that $h \in C(\mathbb{T})$.

We have the following property about the elements of the space \mathcal{C}_{W_α} .

Lemma 3.1. $\mathcal{C}_{W_\alpha} \subseteq C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$.

Proof. Let $H \in \mathcal{C}_{W_\alpha}$, then there exists a function $h \in C(\mathbb{T})$ and $\tilde{a}, \tilde{b} \in \mathbb{R}$ such that H can be written as in (15). Define, for all $u \in [b, 1+b]$,

$$\tilde{H}(u) = \tilde{a} + \tilde{b}u + \int_0^u \int_0^v \tilde{h}(w) dw dv + \frac{1}{\alpha} \left(\tilde{b} + \int_0^b h(w) dw \right),$$

where the function $\tilde{h} : (0, b+1) \rightarrow \mathbb{R}$ coincides with h in $(0, 1]$ and for all $u \in (1, b+1)$, $\tilde{h}(u)$ is equal to $h(u-1)$. We would like prove that $H(u) = \tilde{H}(u)$, for all $u \in (b, 1+b)$. To prove this equality, we need to consider $\mathbb{T} \equiv (0, 1]$. It is clear that $H(u) = \tilde{H}(u)$, for all $u \in (b, 1]$. For $u \in (1, 1+b)$, we use (16) to obtain:

$$\begin{aligned} \tilde{H}(u) &= \tilde{a} + \tilde{b}(u-1) + \tilde{b} + \int_0^1 \int_0^v h(w) dw dv \\ &\quad + \int_1^u \left(\int_0^1 h(w) dw + \int_1^v h(w-1) dw \right) dv + \frac{1}{\alpha} \left(\tilde{b} + \int_0^b h(w) dw \right) \\ &= \tilde{a} + \tilde{b}(u-1) + \int_1^u \int_1^v h(w-1) dw dv. \end{aligned}$$

Changing variables, $\tilde{H}(u)$ can be rewritten as

$$\tilde{a} + \tilde{b}(u-1) + \int_0^{u-1} \int_0^v h(w) dw dv.$$

Since the representation of $u \in (b, b+1)$ in torus is $u-1$ and since for all $u \in (0, b)$

$$H(u) = \tilde{a} + \tilde{b}u + \int_0^u \int_0^v h(w) dw dv,$$

then $\tilde{H}(u) = H(u-1) = H(u)$, for all $u \in (b, b+1)$. \square

Lemma 3.2. *If $H \in \mathcal{C}_{W_\alpha}$, then $\partial_u H(b^+) = \partial_u H(b^-) = \alpha(H(b^+) - H(b^-))$.*

Proof. Since $H \in \mathcal{C}_{W_\alpha}$, a simple computation shows that for $u \in \mathbb{T} \equiv (0, 1]$

$$H(u) = G(u) + \frac{\theta}{\alpha} \mathbf{1}_{[b,1]}(u),$$

where

$$G(u) = \tilde{a} + \tilde{b}u + \int_0^u \int_0^v h(w) dw dv$$

and

$$\theta = \tilde{b} + \int_0^b h(w) dw.$$

Notice that $G(\cdot)$ is continuous and smooth. Then, $H(b^+) = G(b) + \frac{\theta}{\alpha}$ and $H(b^-) = G(b)$. On the other hand,

$$\partial_u H(b^+) = \partial_u H(b^-) = G'(b).$$

Since $G'(b) = \theta$, then $\partial_u H(b^+) = \partial_u H(b^-) = \alpha(H(b^+) - H(b^-))$. This finishes the proof. \square

In our previous work [3], we took $\alpha = 1$ and in the hydrodynamics we proved that $\rho_t(\cdot)$ is a weak solution of (2), which in particular means that

$$\langle \rho_t, H \rangle - \langle \rho_0, H \rangle - \int_0^t \left\langle \rho_s, \frac{d}{du} \frac{d}{dW} H \right\rangle ds = 0, \quad (17)$$

for all $t \in [0, T]$ and all $H \in \mathcal{H}_{W_1}^1$. For the definition of the operator $\frac{d}{du} \frac{d}{dW}$, we refer to [3] and references therein.

Now, we present a result that relates equations (17) and (12). We notice that by Proposition 6.3 of [3], it is enough to verify equation (17) for functions in \mathcal{C}_{W_α} .

Proposition 3.3. *For $H \in \mathcal{C}_{W_\alpha}$, equation (12) coincides with (17).*

Proof. From Lemma 3.1 we know that \mathcal{C}_{W_α} is a subset of $C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$, which is the space of test functions for the integral equation (12). From the previous lemma, for $H \in \mathcal{C}_{W_\alpha}$ the integral equation (12) becomes as

$$\langle \rho_t, H \rangle - \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, h \rangle ds = 0,$$

where $h = \Delta H$. Notice that a function in \mathcal{C}_{W_α} does not depend on time. Now, from the definition of $\frac{d}{du} \frac{d}{dW} H = h$, see [3], we get that

$$\langle \rho_s, \Delta H \rangle = \langle \rho_s, \frac{d}{du} \frac{d}{dW} H \rangle.$$

This finishes the proof. \square

3.2. Characterization of limit points.

Now, we characterize the limit points for $\beta = 1$, accomplished here in an essentially different way from [3]. Recall the definition of $\{\mathbb{Q}_{n, \mu_n}^{\alpha, \beta} : n \geq 1\}$. In order to keep notation simple and since $\beta = 1$ we do not index these measures on β . Let \mathbb{Q}_* be a limit point of $\{\mathbb{Q}_{n, \mu_n}^\alpha : n \geq 1\}$ whose existence is a consequence of Proposition 4.1 of [3]. Assume, without loss of generality, that $\{\mathbb{Q}_{n, \mu_n}^\alpha : n \geq 1\}$ converges to \mathbb{Q}_* , as $n \rightarrow +\infty$. We prove that \mathbb{Q}_* is concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure: $\pi(t, du) = \rho(t, u) du$, whose density $\rho(t, u)$ is a weak solution of (8).

At first we notice that by Proposition 5.6 of [3], \mathbb{Q}_* is concentrated on trajectories absolutely continuous with respect to the Lebesgue measure $\pi_t(du) = \rho(t, u) du$ such that, $\rho(t, \cdot)$ belongs to $L^2(0, T; \mathcal{H}^1(b, 1+b))$. It is well known that the Sobolev space $\mathcal{H}^1(b, 1+b)$ has special properties: all its elements are absolutely continuous functions with bounded variation, see [1], therefore with well defined lateral limits. Such property is inherited by $L^2(0, T; \mathcal{H}^1(b, 1+b))$ in the sense that we can integrate in time the lateral limits.

Let $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$. We begin by claiming that

$$\begin{aligned} \mathbb{Q}_*^\alpha \left[\pi. : \langle \rho_t, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s, \partial_s H_s + \Delta H_s \rangle ds \right. \\ \left. - \int_0^t \{ \rho_s(b^+) \partial_u H_s(b^+) - \rho_s(b^-) \partial_u H_s(b^-) \} ds \right. \\ \left. + \int_0^t \alpha \{ \rho_s(b^+) - \rho_s(b^-) \} \{ H_s(b^+) - H_s(b^-) \} ds = 0, \quad \forall t \in [0, T] \right] = 1. \end{aligned}$$

In order to prove last equality, its enough to show that, for every $\delta > 0$,

$$\begin{aligned} \mathbb{Q}_*^\alpha \left[\pi. : \sup_{0 \leq t \leq T} \left| \langle \rho_t, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s, \partial_s H_s + \Delta H_s \rangle ds \right. \right. \\ \left. \left. - \int_0^t \{ \rho_s(b^+) \partial_u H_s(b^+) - \rho_s(b^-) \partial_u H_s(b^-) \} ds \right. \right. \\ \left. \left. + \int_0^t \alpha \{ \rho_s(b^+) - \rho_s(b^-) \} \{ H_s(b^+) - H_s(b^-) \} ds \right| > \delta \right] = 0. \end{aligned}$$

Since the boundary integrals are not well-defined in $\mathcal{D}([0, T], \mathcal{M})$, we cannot use directly Portmanteau's Theorem. To avoid this technical obstacle, fix $\varepsilon > 0$, which will be taken small later. Let

$$\iota_\varepsilon(u, v) = \begin{cases} \frac{1}{\varepsilon} \mathbf{1}_{(v, v+\varepsilon)}(u), & \text{if } v \in \mathbb{T} \setminus (b - \varepsilon, b), \\ \frac{1}{\varepsilon} \mathbf{1}_{(b-\varepsilon, b)}(u), & \text{if } v \in (b - \varepsilon, b), \end{cases}$$

be an approximation of the identity in the continuous torus. The meaning of such definition is that, for any $v \in \mathbb{T}$ chosen, the interval where $\iota_\varepsilon \neq 0$ never crosses the point $b \in \mathbb{T}$. The convolution of a measure π with ι_ε is defined by $(\pi * \iota_\varepsilon)(v) = \int \iota_\varepsilon(u, v) \pi(du)$. Now, adding and subtracting the convolution of $\rho(t, u)$ with ι_ε , we can bound from above the previous probability by the sum of

$$\begin{aligned} \mathbb{Q}_*^\alpha \left[\pi. : \sup_{0 \leq t \leq T} \left| \langle \rho_t, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s, \partial_s H_s + \Delta H_s \rangle ds \right. \right. \\ \left. \left. - \int_0^t \{ (\rho_s * \iota_\varepsilon)(b^+) \partial_u H_s(b^+) - (\rho_s * \iota_\varepsilon)(b^-) \partial_u H_s(b^-) \} ds \right. \right. \\ \left. \left. + \int_0^t \alpha \{ (\rho_s * \iota_\varepsilon)(b^+) - (\rho_s * \iota_\varepsilon)(b^-) \} \{ H_s(b^+) - H_s(b^-) \} ds \right| > \delta/3 \right], \end{aligned} \quad (18)$$

$$\begin{aligned} \mathbb{Q}_*^\alpha \left[\pi. : \sup_{0 \leq t \leq T} \left| \int_0^t \{ (\rho_s * \iota_\varepsilon)(b^+) \partial_u H_s(b^+) - (\rho_s * \iota_\varepsilon)(b^-) \partial_u H_s(b^-) \} ds \right. \right. \\ \left. \left. - \int_0^t \{ \rho_s(b^+) \partial_u H_s(b^+) - \rho_s(b^-) \partial_u H_s(b^-) \} ds \right| > \delta/3 \right] \end{aligned}$$

and

$$\begin{aligned} \mathbb{Q}_*^\alpha \left[\pi. : \sup_{0 \leq t \leq T} \left| \int_0^t \alpha \{ (\rho_s * \iota_\varepsilon)(b^+) - (\rho_s * \iota_\varepsilon)(b^-) \} \{ H_s(b^+) - H_s(b^-) \} ds \right. \right. \\ \left. \left. - \int_0^t \alpha \{ \rho_s(b^+) - \rho_s(b^-) \} \{ H_s(b^+) - H_s(b^-) \} ds \right| > \delta/3 \right]. \end{aligned}$$

The convolutions above are suitable averages of ρ around the boundary points b^+ and b^- . Therefore, as $\varepsilon \downarrow 0$, the sets inside the two previous probabilities decrease to sets of null probability. It remains to deal with (18). We claim that we can use Portmanteau's Theorem and Proposition A.3 of [3] in order to conclude that (18)

is bounded from above by

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{Q}_{n, \mu_n}^\alpha \left[\pi. : \sup_{0 \leq t \leq T} \left| \langle \rho_t, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s, \partial_s H_s + \Delta H_s \rangle ds \right. \right. \\ \left. - \int_0^t \left\{ (\rho_s * \iota_\varepsilon)(b^+) \partial_u H_s(b^+) - (\rho_s * \iota_\varepsilon)(b^-) \partial_u H_s(b^-) \right\} ds \right. \\ \left. + \int_0^t \alpha \left\{ (\rho_s * \iota_\varepsilon)(b^+) - (\rho_s * \iota_\varepsilon)(b^-) \right\} \left\{ H_s(b^+) - H_s(b^-) \right\} ds \right| > \delta/3 \Bigg]. \end{aligned}$$

Although the functions H_t , H_0 , $\partial_s H_s + \Delta H_s$, $\iota_\varepsilon(\cdot, b^-)$ and $\iota_\varepsilon(\cdot, b^+)$ may not belong to $C(\mathbb{T})$, we can proceed as in Section 6.2 of [3] in order to justify why (18) is bounded from above by the previous expression. Next we outline the main arguments involved in that procedure. Before applying Portmanteau's Theorem, we replace these functions by continuous functions which coincide with the original ones in the torus, except on a small neighborhood of the discontinuity points of H_t , H_0 , $\partial_s H_s + \Delta H_s$, $\iota_\varepsilon(\cdot, b^-)$ and $\iota_\varepsilon(\cdot, b^+)$ and their L^∞ -norm are bounded from above by the L^∞ -norm of the respective original function. By the exclusion rule, the set where we compare this change has small probability. Thus, now we deal with continuous functions and we are able to apply Portmanteau's Theorem and Proposition A.3 of [3]. After using Portmanteau's Theorem, we return to the original functions, by the same arguments as described above. Then, the claim follows.

Considering \mathbb{T}_n embedded in \mathbb{T} , we notice that b_n is the closest site to the left of b and $b_n + 1$ is the closest site to the right of b . Since $(\pi^n * \iota_\varepsilon)(\frac{x}{n}) = \eta^{\varepsilon n}(x)$ for all $x \in \mathbb{T}_n$, where

$$\eta^{\varepsilon n}(x) = \frac{1}{\varepsilon n} \sum_{y=x+1}^{x+\varepsilon n} \eta(y),$$

by the definition of $\mathbb{Q}_{n, \mu_n}^\alpha$, we can rewrite the previous probability as

$$\begin{aligned} \mathbb{P}_{\mu_n}^\alpha \left[\sup_{0 \leq t \leq T} \left| \langle \pi_t^n, H_t \rangle - \langle \pi_0^n, H_0 \rangle - \int_0^t \langle \pi_s^n, \partial_s H_s + \Delta H_s \rangle ds \right. \right. \\ \left. - \int_0^t \left\{ \eta_s^{\varepsilon n}(b_n + 1) \partial_u H_s(b^+) - \eta_s^{\varepsilon n}(b_n) \partial_u H_s(b^-) \right\} ds \right. \\ \left. + \int_0^t \alpha \left\{ \eta_s^{\varepsilon n}(b_n + 1) - \eta_s^{\varepsilon n}(b_n) \right\} \left\{ H_s(b^+) - H_s(b^-) \right\} ds \right| > \delta/3 \Bigg]. \end{aligned}$$

The next step is to sum and subtract $\int_0^t n^2 \mathcal{L}_n \langle \pi_s^n, H_s \rangle ds$ to the term inside the supremum above and the previous probability becomes bounded from above by the sum of

$$\mathbb{P}_{\mu_n}^\alpha \left[\sup_{0 \leq t \leq T} \left| \langle \pi_t^n, H_t \rangle - \langle \pi_0^n, H_0 \rangle - \int_0^t \langle \pi_s^n, \partial_s H_s \rangle + n^2 \mathcal{L}_n \langle \pi_s^n, H_s \rangle ds \right| > \delta/6 \right]$$

and

$$\begin{aligned} \mathbb{P}_{\mu_n}^\alpha \left[\sup_{0 \leq t \leq T} \left| \int_0^t n^2 \mathcal{L}_n \langle \pi_s^n, H_s \rangle ds - \int_0^t \langle \pi_s^n, \Delta H_s \rangle ds \right. \right. \\ \left. - \int_0^t \left\{ \eta_s^{\varepsilon n}(b_n + 1) \partial_u H_s(b^+) - \eta_s^{\varepsilon n}(b_n) \partial_u H_s(b^-) \right\} ds \right. \\ \left. + \int_0^t \alpha \{ \eta_s^{\varepsilon n}(b_n + 1) - \eta_s^{\varepsilon n}(b_n) \} \{ H_s(b^+) - H_s(b^-) \} ds \right| > \delta/6 \Bigg]. \end{aligned}$$

The expression inside the first probability above, is the martingale associated to the process $\langle \pi_t^n, H_t \rangle$ that we denote by $\mathcal{M}_t^n(H)$. A simple computation shows that $\mathcal{M}_t^n(H)$ converges to zero in $L^2(\mathbb{P}_{\mu_n}^\alpha)$ as $n \rightarrow +\infty$. Then, by Doob's inequality, the first probability vanishes as $n \rightarrow +\infty$, for every $\delta > 0$. Now we treat the remaining term. Using the expression for $n^2 \mathcal{L}_n \langle \pi_s^n, H_s \rangle$, we can bound the previous expression by the sum of

$$\mathbb{P}_{\mu_n}^\alpha \left[\sup_{0 \leq t \leq T} \left| \int_0^t \langle \pi_s^n, \Delta H_s \rangle ds - \int_0^t \frac{1}{n} \sum_{x \neq b_n, b_n+1} \eta_s(x) \Delta_n H_s(x/n) ds \right| > \delta/18 \right],$$

$$\begin{aligned} \mathbb{P}_{\mu_n}^\alpha \left[\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \eta_s^{\varepsilon n}(b_n + 1) \partial_u H_s(b^+) - \eta_s^{\varepsilon n}(b_n) \partial_u H_s(b^-) \right\} ds \right. \right. \\ \left. - \int_0^t \left\{ \eta_s(b_n + 1) n \nabla_n H_s(b_n + 1) - \eta_s(b_n) n \nabla_n H_s(b_n - 1) \right\} ds \right| > \delta/18 \Bigg] \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}_{\mu_n}^\alpha \left[\sup_{0 \leq t \leq T} \left| \int_0^t \alpha \{ \eta_s^{\varepsilon n}(b_n + 1) - \eta_s^{\varepsilon n}(b_n) \} \{ H_s(b^+) - H_s(b^-) \} ds \right. \right. \\ \left. - \int_0^t \alpha \{ \eta_s(b_n + 1) - \eta_s(b_n) \} \nabla_n H_s(b_n) ds \right| > \delta/18 \Bigg], \end{aligned}$$

where for $x \in \mathbb{T}_n$, $\Delta_n H(x/n) = n^2 (H(\frac{x+1}{n}) + H(\frac{x-1}{n}) - 2H(\frac{x}{n}))$ and $\nabla H(x/n) = n(H((x+1)/n) - H(x/n))$. Since $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$, the discrete Laplacian of H_s converges uniformly to the continuous Laplacian of H_s and the first probability is null.

To prove that the remaining probabilities are null, we observe that $n \nabla_n H_s$ converges uniformly to $\partial_u H_s$, as $n \rightarrow +\infty$ and $\nabla_n H_s(b_n)$ converges uniformly to $H_s(b^+) - H_s(b^-)$, as $n \rightarrow +\infty$, since $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$. By the exclusion constrain and approximating the integral by Riemannian sums, the previous probabilities vanish as long as we show that

$$\begin{aligned} \mathbb{P}_{\mu_n}^\alpha \left[\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \eta_s^{\varepsilon n}(b_n + 1) - \eta_s(b_n + 1) \right\} \partial_u H_s(b^+) \right. \right. \\ \left. - \left\{ \eta_s^{\varepsilon n}(b_n) - \eta_s(b_n) \right\} \partial_u H_s(b^-) ds \right| > \delta \Bigg] \end{aligned}$$

and

$$\mathbb{P}_{\mu_n}^\alpha \left[\sup_{0 \leq t \leq T} \left| \int_0^t \alpha \left\{ \left\{ \eta_s^{\varepsilon_n}(b_n + 1) - \eta_s^{\varepsilon_n}(b_n) \right\} - \left\{ \eta_s(b_n + 1) - \eta_s(b_n) \right\} \right\} \left\{ H_s(b^+) - H_s(b^-) \right\} ds \right| > \delta \right].$$

converge to zero, as $\varepsilon \downarrow 0$, $\forall \delta > 0$, respectively. This is a consequence of Lemma 5.4 of [3].

4. UNIQUENESS OF WEAK SOLUTIONS

We present here the proof of uniqueness of weak solutions of (7), (8) and (10). Since all the equations are linear, it is sufficient to consider the initial condition $\rho_0(\cdot) \equiv 0$. We start by showing uniqueness of weak solutions of (8) and later we present a simpler proof than the one presented in [3] for both equations (7) and (10).

4.1. Uniqueness of weak solutions of (8).

To simplify notation, along this subsection, we consider $b^+ = b$ and $b^- = 1 + b$. We choose this notation in order to identify the torus \mathbb{T} with the interval $[b, 1 + b)$. Denote by $L^2(\mathbb{T})^{\perp 1}$ the subspace of functions $g \in L^2(\mathbb{T})$ with zero mean, namely:

$$\int_{\mathbb{T}} g(u) du = 0.$$

Definition 12. Denote by $\mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$ the space of functions $H : \mathbb{T} \rightarrow \mathbb{R}$ that satisfy

- H is twice differentiable;
- $\partial_u H$ is absolutely continuous on $\mathbb{T} \setminus \{b\}$;
- $\Delta H \in L^2(\mathbb{T})^{\perp 1}$;
- H satisfies the boundary conditions:

$$\partial_u H(b) = \partial_u H(1 + b) = \alpha(H(b) - H(1 + b)). \quad (19)$$

Proposition 4.1. Let $\rho : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ be a weak solution of (8). Then, for all $t \in [0, T]$ and for all $H \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$, it holds that

$$\langle \rho_t, H \rangle - \langle \rho_0, H \rangle = \int_0^t \langle \rho_s, \Delta H \rangle ds. \quad (20)$$

Proof. Fix $H \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$. Let $\{h_n : n \geq 1\} \subset C^2(\mathbb{T})$ be such that for all $n \geq 1$ $\int_{\mathbb{T}} h_n(u) du = 0$ and $\{h_n : n \geq 1\}$ converges in $L^2(\mathbb{T})$ to ΔH , as $n \rightarrow +\infty$. Also, take $\{\beta_n : n \geq 1\} \subset \mathbb{R}$ converging to $\partial_u H(b)$, as $n \rightarrow +\infty$. Define, for each $u \in [b, 1 + b]$,

$$\tilde{H}_n(u) = H(b) + \beta_n(u - b) + \int_b^u \int_b^v h_n(w) dw dv.$$

For each $u \in \mathbb{T}$, denote $H_n(u) := \tilde{H}_n(u)$. Notice that we are identifying \mathbb{T} with $[b, 1 + b)$. Thus for each $n \geq 1$, $H_n \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$. Since H satisfies (12)

and since for each $n \geq 1$ $\partial_u H_n(b) = \partial_u H_n(1+b) = \beta_n$ and $\Delta H_n = h_n$, then

$$\begin{aligned} \langle \rho_t, H_n \rangle - \langle \rho_0, H_n \rangle &= \int_0^t \langle \rho_s, h_n \rangle ds + \int_0^t \{ \rho_s(b) - \rho_s(1+b) \} \partial_u H_n(b) ds \\ &\quad - \int_0^t \alpha \{ \rho_s(b) - \rho_s(1+b) \} \{ H_n(b) - H_n(1+b) \} ds. \end{aligned}$$

Observe that H does not depend on time.

Now, using that H_n converges to H as $n \rightarrow +\infty$, h_n converges to ΔH as $n \rightarrow +\infty$ and $\partial_u H_n(b) = \partial_u H_n(1+b) = \beta_n$ converges to $\partial_u H(b)$ as $n \rightarrow +\infty$, we get that $H_n(1+b)$ converges to

$$H(b) + \partial_u H(b) + \int_b^1 \int_b^v \Delta H(w) dw dv,$$

as $n \rightarrow +\infty$. By definition of $\mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$, that last expression is equal to $H(1+b)$. Thus, $H_n(b) - H_n(1+b)$ converges to $H(b) - H(1+b)$, as $n \rightarrow +\infty$. Therefore, sending $n \rightarrow +\infty$ in the previous equality, we arrive at

$$\begin{aligned} \langle \rho_t, H \rangle - \langle \rho_0, H \rangle &= \int_0^t \langle \rho_s, \Delta H \rangle ds + \int_0^t \{ \rho_s(b) - \rho_s(1+b) \} \partial_u H(b) ds \\ &\quad - \int_0^t \alpha \{ \rho_s(b) - \rho_s(1+b) \} \{ H(b) - H(1+b) \} ds. \end{aligned}$$

By (19) the proof ends. \square

The next step is to construct the inverse of the operator $\Delta : \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T}) \rightarrow L^2(\mathbb{T})$. For $g \in L^2(\mathbb{T})^{\perp 1}$, define

$$[(-\Delta)_\alpha^{-1}g](u) = \int_{\mathbb{T}} G_\alpha(u, w) g(w) dw,$$

where the function $G_\alpha : [b, 1+b] \times [b, 1+b] \rightarrow \mathbb{R}$ is given by

$$G_\alpha(u, w) = \frac{\alpha}{\alpha+1} u(1-w) - (u-w) \mathbf{1}_{\{b \leq w \leq u \leq 1+b\}},$$

and \mathbb{T} is identified with $[b, 1+b]$. Then, for $g \in L^2(\mathbb{T})^{\perp 1}$:

$$[(-\Delta)_\alpha^{-1}g](u) = \frac{\alpha}{\alpha+1} u \int_{\mathbb{T}} (1-w)g(w) dw - u \int_b^u g(w) dw + \int_b^u w g(w) dw. \quad (21)$$

Proposition 4.2. *The operator $(-\Delta)_\alpha^{-1}$ enjoys the following properties:*

- (a) $\forall g \in L^2(\mathbb{T})^{\perp 1}$, $(-\Delta)_\alpha^{-1}g \in C^1(\mathbb{T} \setminus \{b\})$ and $\partial_u [(-\Delta)_\alpha^{-1}g]$ is absolutely continuous in $\mathbb{T} \setminus \{b\}$, both having finite side limits at the point $b \in \mathbb{T}$ as on (9);
- (b) $\forall g \in L^2(\mathbb{T})^{\perp 1}$,
- $\partial_u [(-\Delta)_\alpha^{-1}g](b) = \partial_u [(-\Delta)_\alpha^{-1}g](1+b) = \alpha \left([(-\Delta)_\alpha^{-1}g](b) - [(-\Delta)_\alpha^{-1}g](1+b) \right);$
- (c) $\forall g \in L^2(\mathbb{T})^{\perp 1}$, $(-\Delta)_\alpha^{-1}g \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$;
- (d) $\forall g \in L^2(\mathbb{T})^{\perp 1}$, $(-\Delta)[(-\Delta)_\alpha^{-1}g] = g$;
- (e) The operators $(-\Delta) : \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T}) \rightarrow L^2(\mathbb{T})^{\perp 1}$ and $(-\Delta)_\alpha^{-1} : L^2(\mathbb{T})^{\perp 1} \rightarrow \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$ are symmetric and non-negative;
- (f) $\forall g \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$, $\int_{\mathbb{T}} \Delta g(u) du = 0$.

Proof. Easily by (21), one obtains (a). Item (b) follows from assumption $g \in L^2(\mathbb{T})^{\perp 1}$. The items (a) and (b) imply (c). Deriving (21) twice and recalling item (c), we obtain (d).

Now, fix $g, h \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$ and by integration by parts:

$$\langle -\Delta g, h \rangle = \langle \partial_u g, \partial_u h \rangle + \partial_u g(b)h(b) - \partial_u g(1+b)h(1+b).$$

Since $g, h \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$, then these functions satisfy (19) and as a consequence we obtain that

$$\langle -\Delta g, h \rangle = \langle \partial_u g, \partial_u h \rangle + \frac{1}{\alpha} \partial_u g(b) \partial_u h(b),$$

which implies symmetry and non-negativity of Δ . The same argument applies for $(-\Delta)_\alpha^{-1}$, by item (d). Item (f) follows from last equality by taking $h = -1$. \square

Proposition 4.3. *Let ρ be a weak solution of (8) with $\rho_0 \equiv 0$. Then, for all $t \in [0, T]$, it holds that*

$$\langle \rho_t, (-\Delta)_\alpha^{-1} \rho_t \rangle = -2 \int_0^t \langle \rho_s, \rho_s \rangle ds. \quad (22)$$

In particular, since equation (8) is linear, there is at most one weak solution with initial condition ρ_0 .

Proof. Notice at first that $\rho_t \in L^2(\mathbb{T})^{\perp 1}$ for any time $t \in [0, T]$. Take a partition $0 = t_0 < t_1 < \dots < t_n = t$ of the interval $[0, t]$ so that

$$\langle \rho_t, (-\Delta)_\alpha^{-1} \rho_t \rangle - \langle \rho_0, (-\Delta)_\alpha^{-1} \rho_0 \rangle = \sum_{k=0}^{n-1} \langle \rho_{t_{k+1}}, (-\Delta)_\alpha^{-1} \rho_{t_{k+1}} \rangle - \langle \rho_{t_k}, (-\Delta)_\alpha^{-1} \rho_{t_k} \rangle.$$

Therefore, we have to estimate

$$\begin{aligned} & \sum_{k=0}^{n-1} \langle \rho_{t_{k+1}}, (-\Delta)_\alpha^{-1} \rho_{t_{k+1}} \rangle - \langle \rho_{t_{k+1}}, (-\Delta)_\alpha^{-1} \rho_{t_k} \rangle \\ & + \sum_{k=0}^{n-1} \langle \rho_{t_{k+1}}, (-\Delta)_\alpha^{-1} \rho_{t_k} \rangle - \langle \rho_{t_k}, (-\Delta)_\alpha^{-1} \rho_{t_k} \rangle. \end{aligned} \quad (23)$$

First we estimate the term on the left hand side of the previous equation. The remaining term can be estimated in a similar way because $(-\Delta)_\alpha^{-1}$ is a symmetric operator.

Recalling that $\rho_{t_k} \in L^2(\mathbb{T})^{\perp 1}$, Proposition 4.2 item (c) and Proposition 4.1 we get that

$$\langle \rho_{t_{k+1}}, (-\Delta)_\alpha^{-1} \rho_{t_k} \rangle - \langle \rho_{t_k}, (-\Delta)_\alpha^{-1} \rho_{t_k} \rangle = - \int_{t_k}^{t_{k+1}} \langle \rho_s, \rho_s \rangle ds + \int_{t_k}^{t_{k+1}} \langle \rho_s, \rho_s - \rho_{t_k} \rangle ds.$$

The sum over k of the first term on the right side of last equality equals to $-\int_0^t \langle \rho_s, \rho_s \rangle ds$. We get the same term by treating the remaining term of (23), which finishes the proof.

Now, we shall treat the remainder. Let $\iota_\delta : \mathbb{T} \rightarrow \mathbb{R}$ be a smooth approximation of the identity and $\Phi_\delta : \mathbb{T} \rightarrow \mathbb{R}$ a smooth function bounded by one, equal to zero in the interval $(-\delta, \delta)$ and equals to one in $\mathbb{T} \setminus (-2\delta, 2\delta)$. Let $\rho_s^\delta(u) = (\rho_s * \iota_\delta)(u) \Phi_\delta(u)$. It is of easy verification that $\rho_s^\delta \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$, for any $s \in [0, T]$, and also that $\rho_s^\delta(\cdot)$

converges to $\rho_s(\cdot)$ in $L^2(\mathbb{T})$ when $\delta \downarrow 0$. Adding and subtracting ρ^δ , the second term on the right hand side of last equality can be written as

$$\int_{t_k}^{t_{k+1}} \langle \rho_s - \rho_s^\delta, \rho_s - \rho_{t_k} \rangle ds + \int_{t_k}^{t_{k+1}} \langle \rho_s^\delta, \rho_s - \rho_{t_k} \rangle ds.$$

Fix $\varepsilon > 0$. Since $\rho_s^\delta(\cdot)$ converges to $\rho_s(\cdot)$ in $L^2(\mathbb{T})$, applying the Dominated Convergence Theorem, the absolute value of the sum in k of the term on the left hand side of the previous expression is bounded from above by ε for some $\delta(\varepsilon)$ small. Take $\delta = \delta(\varepsilon)$. Since $\rho_s^\delta \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$ and since ρ is a weak solution of (8), the second term in the previous expression is equal to

$$\int_{t_k}^{t_{k+1}} \int_{t_k}^s \langle \rho_r, \Delta \rho_s^\delta \rangle dr ds,$$

whose absolute value is bounded from above by $C(\rho, \delta)(t_{k+1} - t_k)^2$, concluding the proof of the first claim.

Now we prove the second claim. Since $\rho_t \in \mathbb{L}^2(\mathbb{T})^\perp$, then by (e) of Proposition 4.2, $\langle \rho_t, (-\Delta)_\alpha^{-1} \rho_t \rangle \geq 0$, for all $t \in [0, T]$. But from (22) we conclude that $\langle \rho_t, (-\Delta)_\alpha^{-1} \rho_t \rangle = 0$, for all $t \in [0, T]$. From item (d), fixed $t \in [0, T]$, there exists $f_t \in \mathcal{H}_{bc}^{2,\alpha}(\mathbb{T})$ such that $\rho_t = (-\Delta)f_t$, and thus

$$\langle \rho_t, (-\Delta)_\alpha^{-1} \rho_t \rangle = \langle -\Delta f_t, f_t \rangle = \langle \partial_u f_t, \partial_u f_t \rangle + \frac{1}{\alpha} (\partial_u f_t(b))^2.$$

Then, $\partial_u f_t(u) = 0$, u - almost surely and for all $t \in [0, T]$. Since $\rho_t = (-\Delta)f_t$, we have $\rho_t(u) = 0$, u - almost surely and for all $t \in [0, T]$. This concludes the proof. \square

4.2. Uniqueness of weak solutions of (7) and (10).

Let ρ_t be a weak solution of (10) with $\rho_0 \equiv 0$ and to simplify notation we consider $b = 0$. For $u \in \mathbb{T}$, let $H_k(u) = \sqrt{2} \cos(k\pi u)$, $k \in \mathbb{N}$. Recalling (13), for all $k \in \mathbb{N}$, $H_k \in C^2(\mathbb{T} \setminus \{0\})$ and $\partial_u H_k(0^+) = \partial_u H_k(0^-) = 0$, we have that

$$\langle \rho_t, H_k \rangle = -(k\pi)^2 \int_0^t \langle \rho_s, H_k \rangle ds.$$

Now, by Gronwall's inequality it follows that $\langle \rho_t, H_k \rangle = 0$, $\forall t > 0$ and for all $k \in \mathbb{N}$. Since $\{H_k; k \in \mathbb{N}\}$ is a complete orthonormal system in $L^2(\mathbb{T})$, we obtain that $\rho_t \equiv 0$, $\forall t > 0$.

The proof of uniqueness of weak solutions of (7) is the same as above, but one has to consider the complete orthonormal system

$$\{1, \sqrt{2} \cos(2k\pi u), \sqrt{2} \sin(2k\pi u); k \in \mathbb{N}\}$$

composed of functions in $C^2(\mathbb{T})$.

5. PROOF OF THEOREM 2.2

The next proposition is the connection between the particle system $\{\eta_t; t \geq 0\}$ and Theorem 2.2. As in Section 3.2 since we are restricted to $\beta = 1$ we omit the dependence on this parameter in what follows.

Proposition 5.1. *Let \mathbb{Q}_*^α be a limit point of $\{\mathbb{Q}_{n, \mu_n}^\alpha : n \geq 1\}$. Then,*

$$\mathbb{E}_{\mathbb{Q}_*^\alpha} \left[\sup_{H \in C^{0,1}([0,T] \times \mathbb{T})} \left\{ \langle \rho, \partial_u H \rangle - 2 \langle H, H \rangle_{W_\alpha} \right\} \right] \leq K_0.$$

The proof of this proposition follows the same lines of [3, Subsection 5.2]. Here we point out only the differences from the case proved in [3].

For a function $H \in C^{0,1}([0, T] \times \mathbb{T})$, $\delta > 0$, $\varepsilon > 0$, $n \geq 1$ and $\eta \in \{0, 1\}^{\mathbb{T}_n}$, let

$$\begin{aligned} U_n(\varepsilon, \delta, H, \eta) &= \frac{1}{\varepsilon n} \sum_{x \in \mathbb{T}_n} H\left(\frac{x}{n}\right) \{\eta(x) - \eta(x + \varepsilon n)\} \\ &\quad - \frac{2}{n} \sum_{x \in \mathbb{T}_n} \left(H\left(\frac{x}{n}\right)\right)^2 \left\{1 - \frac{1}{\alpha \varepsilon} \mathbf{1}_{[b-\varepsilon, b)}\left(\frac{x}{n}\right)\right\}. \end{aligned}$$

Lemma 5.2. *Fix $H \in C^{0,1}([0, T] \times \mathbb{T})$ and $\gamma \in (0, 1)$. Recall that ν_γ^n denotes the Bernoulli product measure on $\{0, 1\}^{\mathbb{T}_n}$. Let f be a density with respect to ν_γ^n . Then,*

$$\int U_n(\varepsilon, \delta, H, \eta) f(\eta) \nu_\gamma^n(d\eta) \leq n \mathcal{D}_n(f),$$

where $\mathcal{D}_n(f) = \langle -\mathcal{L}_n \sqrt{f}, \sqrt{f} \rangle_{\nu_\gamma^n}$ is the Dirichlet form of f with respect to ν_γ^n .

Proof. Since it is very similar to the proof of Lemma 5.5 in [3] we omitted it. \square

Lemma 5.3. *Consider a dense sequence $\{H_\ell : \ell \geq 1\}$ in $C^{0,1}([0, T] \times \mathbb{T})$. For every $k \geq 1$, and every $\varepsilon > 0$,*

$$\overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow +\infty} \mathbb{E}_{\mu_n}^\alpha \left[\max_{1 \leq i \leq k} \left\{ \int_0^T U_n(\varepsilon, \delta, H_i(s, \cdot), \eta_s) ds \right\} \right] \leq K_0,$$

where K_0 is such that $H(\mu_n | \nu_\gamma^n) \leq K_0 n$ and $H(\mu_n | \nu_\gamma^n)$ is the entropy of μ_n with respect to ν_γ^n .

Proof. This proof is a consequence of Feynman–Kac formula and Lemma 5.2. Since it is similar to the proof of Lemma 5.8 in [3] we omitted it. \square

As in the beginning of Section 3.2, we assume without loss of generality that the sequence $\mathbb{Q}_{n, \mu_n}^\alpha$ converges to \mathbb{Q}_*^α , as $n \rightarrow +\infty$.

Proof of Proposition 5.1. It follows by Lemma 5.3, in the same way as we did in the proof of Lemma 5.7 in [3]. \square

Proof of Theorem 2.2. To prove the theorem we notice at first that, the existence of weak solutions of equation (8) is granted by tightness together with the characterization of limit points given in Section 3.2. The uniqueness of weak solutions of (8) is proved in Section 4.1. Finally, by noticing the characterization of \mathbb{Q}_*^α given in Section 3.2, Proposition 5.1 concludes the proof. \square

6. PROOF OF THEOREM 2.3.

This section is devoted to the proof of Theorem 2.3. We notice that since we have already proved Theorem 2.2, from now on, for fixed $\alpha > 0$, we denote by $\rho^\alpha : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ the unique weak solution of (8).

The proof follows several steps that we announce as follows. Firstly, in Proposition 6.1 we prove that the sequence $\{\rho^\alpha : \alpha > 0\}$ is bounded in $L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$. Secondly, in Proposition 6.2, we prove that, any limit of a convergent subsequence of $\{\rho^{\alpha_n} : n \geq 1\}$ is in $L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$. Then, in Proposition 6.3, we obtain some information on the weak solution of (8). Then, the next step is to analyze separately each term of equation (12) and to obtain asymptotic results for its terms as stated in Proposition 6.4 and Proposition 6.5. Finally, in Proposition 6.6 and

Proposition 6.7 we prove the final step in order to conclude the proof of the Theorem.

In order to keep notation simple, we denote by $C_c(b)$ the space of functions $H \in C^{0,1}([0, T] \times \mathbb{T})$ with compact support in $[0, T] \times (\mathbb{T} \setminus \{b\})$.

Proposition 6.1. *The sequence $\{\rho^\alpha : \alpha > 0\}$ is bounded in $L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$.*

Proof. We begin by observing that Theorem 2.2 implies the inequality

$$\langle\langle \rho^\alpha, \partial_u H \rangle\rangle - 2\langle\langle H, H \rangle\rangle \leq K_0, \quad (24)$$

for all $H \in C_c(b)$. This result is a consequence of H vanishing at b which implies that $\langle\langle H, H \rangle\rangle = \langle\langle H, H \rangle\rangle_{W_\alpha}$. Now, under inequality (24) and recalling that K_0 does not depend on α , since

$$\sup_{H \in C_c(b)} \left\{ \langle\langle \rho^\alpha, \partial_u H \rangle\rangle - 2\langle\langle H, H \rangle\rangle \right\} = \frac{1}{8} \int_0^T \|\partial_u \rho_t^\alpha\|_{L^2(\mathbb{T})}^2 dt,$$

which is proved in Corollary A.5, we conclude that $\{\rho^\alpha; \alpha > 0\}$ is bounded in $L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$. \square

The boundedness of $\{\rho^\alpha : \alpha > 0\}$ in $L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$ implies a compact embedding of $\{\rho^\alpha : \alpha > 0\}$ in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$. This is a particular case of the Rellich-Kondrachov's Theorem for spaces involving time, that can be found in [7]. To verify it in detail, we list the steps: following the notation of [7, Page 271, Subsection 2.2], take $X_0 = X = \mathcal{H}^1(\mathbb{T} \setminus \{b\})$, $X_1 = L^2(\mathbb{T} \setminus \{b\})$ and notice that any Hilbert space is reflexive. This attains the hypothesis of [7, Theorem 2.1, page 271] and corresponds to the case we consider. By this compact embedding, any sequence $\{\rho^{\alpha_n} : n \geq 1\}$ has a convergent subsequence in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$.

Next, we show that the limit of a convergent subsequence of $\{\rho^\alpha : \alpha > 0\}$ is in the space $L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$.

Proposition 6.2. *If ρ^* is the limit in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$ of some sequence in the set $\{\rho^\alpha : \alpha > 0\}$, then $\rho^* \in L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$.*

Proof. Suppose that ρ^{α_n} converges to ρ^* in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$, as $n \rightarrow +\infty$. By Theorem 2.2, for each $n \geq 1$, ρ^{α_n} satisfies (24) for any $H \in C_c(b)$:

$$\int_0^T \langle \rho_s^{\alpha_n}, \partial_u H_s \rangle ds - 2 \int_0^T \langle H_s, H_s \rangle ds \leq K_0$$

and K_0 does not depend neither on n nor H . Sending $n \rightarrow +\infty$ in the previous inequality, we get that

$$\int_0^T \langle \rho_s^*, \partial_u H_s \rangle ds - 2 \int_0^T \langle H_s, H_s \rangle ds \leq K_0.$$

Replacing H by cH in the previous inequality, and minimizing over $c \in \mathbb{R}$, gives that

$$\begin{aligned} \varphi : C_c(b) &\rightarrow \mathbb{R} \\ H &\mapsto \int_0^T \langle \rho_s^*, \partial_u H_s \rangle ds \end{aligned}$$

is a bounded linear functional. Notice that $C_c(b)$ is a dense in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$. Hence, by Riesz Representation Theorem, there exists $\partial_u \rho^* \in L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$ such that

$$\int_0^T \langle \rho_s^*, \partial_u H_s \rangle ds = - \int_0^T \langle \partial_u \rho_s^*, H_s \rangle ds ,$$

for all functions $H \in C_c(b)$. That is ρ^* belongs to $L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$. This finishes the proof. \square

By integration by parts, equation (12) can be rewritten as

$$\begin{aligned} & \langle \rho_t^\alpha, H_t \rangle - \langle \rho_0^\alpha, H_0 \rangle + \int_0^t \langle \partial_u \rho_s^\alpha, \partial_u H_s \rangle ds - \int_0^t \langle \rho_s^\alpha, \partial_s H_s \rangle ds \\ & + \int_0^t \alpha \{ \rho_s^\alpha(b^+) - \rho_s^\alpha(b^-) \} \{ H_s(b^+) - H_s(b^-) \} ds = 0 , \end{aligned} \quad (25)$$

where $\partial_u \rho^\alpha$ is the weak derivative of ρ^α . Our goal now, consists in analyzing the limit as $\alpha \rightarrow 0$ or $\alpha \rightarrow \infty$ of the terms in the previous equation. Due to boundary restrictions, last integral term in (25) is analyzed separately. Moreover, Proposition 6.6 covers the case $\alpha \rightarrow 0$ and Proposition 6.7 covers the case $\alpha \rightarrow \infty$. We begin by showing some smoothness of a weak solution of (8).

Proposition 6.3. *For any $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$ there exists a constant C_H^T not depending on α such that*

$$| \langle \rho_t^\alpha, H_t \rangle - \langle \rho_s^\alpha, H_s \rangle | \leq C_H^T |t - s|^{1/2}, \quad \forall s, t \in [0, T].$$

Proof. Let $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$. Since ρ^α satisfies (25), we have to estimate the absolute value of:

$$\begin{aligned} R_1 &:= \int_s^t \langle \partial_u \rho_r^\alpha, \partial_u H_r \rangle dr , \\ R_2 &:= \int_s^t \langle \rho_r^\alpha, \partial_r H_r \rangle dr , \\ R_3 &:= \int_s^t \alpha \{ \rho_r^\alpha(b^+) - \rho_r^\alpha(b^-) \} \{ H_r(b^+) - H_r(b^-) \} dr . \end{aligned}$$

We start by the case $\alpha \geq 1$. At first we notice that Proposition A.6 guarantees that R_3 can be rewritten as

$$\int_s^t \partial_u \rho_r^\alpha(b) \{ H_r(b^+) - H_r(b^-) \} dr .$$

By Cauchy-Schwarz inequality, we have that

$$|R_3| \leq \left(\int_0^T (\partial_u \rho_r^\alpha(b))^2 dr \right)^{1/2} 2 \|H\|_\infty |t - s|^{1/2} ,$$

where

$$\|F\|_\infty := \sup_{(t,u) \in [0,T] \times \mathbb{T}} |F(t,u)| .$$

By Theorem 2.2 and since $\alpha \geq 1$, the function ρ^α satisfies

$$\langle \rho^\alpha, \partial_u H \rangle - 2 \langle H, H \rangle_{W_1} \leq K_0 ,$$

for all $H \in C^{0,1}([0, T] \times \mathbb{T})$. Thus, by Proposition A.4 we conclude that

$$\int_0^T (\partial_u \rho_r^\alpha(b))^2 dr \leq 8K_0.$$

We arrive at $|R_3| \leq (8K_0)^{1/2} 2\|H\|_\infty |t-s|^{1/2}$. Notice that by Cauchy-Schwarz inequality, Theorem 2.2 and Proposition A.4, we get that

$$|R_1| \leq (8K_0)^{1/2} 2\|\partial_u H\|_\infty |t-s|^{1/2}.$$

Finally, R_2 can be easily bounded from above by $\|\partial_r H\|_\infty |t-s|$.

Now we treat the case $\alpha < 1$. Since to estimate R_1 and R_2 we did not impose any restriction in α , it remains to estimate R_3 which is easily bounded from above by $4\|H\|_\infty |t-s|$. To conclude it is enough to estimate $|t-s|$ by $(2T)^{1/2}|t-s|^{1/2}$. \square

Proposition 6.4. *Suppose that ρ^{α_n} converges to ρ^* in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$, as $n \rightarrow +\infty$. Then, for all $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$,*

$$\lim_{n \rightarrow +\infty} \langle \rho_t^{\alpha_n}, H_t \rangle = \langle \rho_t^*, H_t \rangle,$$

for $t \in [0, T]$ almost surely. Moreover, there exists a function $\tilde{\rho}$ such that $\rho^* = \tilde{\rho}$ almost surely and $t \mapsto \langle \tilde{\rho}_t, H_t \rangle$ is continuous.

Proof. The Dominated Convergence Theorem implies that $\rho_t^{\alpha_n}$ converges to ρ_t^* in $L^2(\mathbb{T})$, as $n \rightarrow +\infty$, for $t \in [0, T]$ almost surely. Cauchy-Schwarz inequality implies that

$$|\langle \rho_t^{\alpha_n}, H_t \rangle - \langle \rho_t^*, H_t \rangle| \leq \|\rho_t^{\alpha_n} - \rho_t^*\|_{L^2(\mathbb{T})} \|H_t\|_{L^2(\mathbb{T})}$$

which together with the previous observation finishes the proof of the first statement.

For the second statement, fix $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$ and for each $n \geq 1$ consider the function $f_n(\cdot, H) : [0, T] \rightarrow \mathbb{R}$ given by $f_n(t, H) := \langle \rho_t^{\alpha_n}, H_t \rangle$. By Proposition 6.3 it follows that $\{f_n(\cdot, H) : n \geq 1\}$ is equicontinuous. Since $|f_n(t, H)| \leq \|H\|_\infty$, by Arzelà-Ascoli Theorem, there exists a subsequence n_k , depending on H , such that $f_{n_k}(\cdot, H)$ converges uniformly in t , as $k \rightarrow +\infty$, to $f(\cdot, H)$ which is continuous.

Notice that for fixed $t \in [0, T]$ and $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$, $f(t, H)$ is a function of $H_t \in C^2(\mathbb{T} \setminus \{b\})$, that we denote by $g(H_t)$. To check that g is a bounded linear functional defined in $L^2(\mathbb{T})$ we do the following. Since $C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$ is separable, applying a diagonal argument we can find a subsequence n_j , which is uniformly on H , such that the convergence above holds uniformly in t , along n_j , for any function in a countable dense set of $C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$. Then g is a bounded linear functional in $C^2(\mathbb{T} \setminus \{b\})$, which can be extended to a bounded linear functional in $L^2(\mathbb{T})$.

Now, Riesz Representation Theorem implies the existence of a function $\tilde{\rho}_t \in L^2(\mathbb{T})$ such that $g(H_t) = \langle \tilde{\rho}_t, H_t \rangle$. Notice that last equality holds for all $t \in [0, T]$. For $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$, $\langle \tilde{\rho}_t, H_t \rangle = g(H_t) = \lim_{j \rightarrow +\infty} \langle \rho_t^{\alpha_{n_j}}, H_t \rangle$, for all $t \in [0, T]$. Thus, $\tilde{\rho}_t = \rho_t^*$, $t \in [0, T]$ almost surely, because $\langle \rho_t^{\alpha_{n_j}} - \tilde{\rho}_t, H_t \rangle$ and $\langle \rho_t^{\alpha_{n_j}} - \rho_t^*, H_t \rangle$ go to zero, as $j \rightarrow +\infty$. \square

From now on, as a consequence of the previous proposition, we consider $\tilde{\rho}$ instead of ρ^* , but we keep the same notation.

Proposition 6.5. *Suppose that ρ^{α_n} converges to ρ^* in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$, as $n \rightarrow +\infty$. Then, for all $t \in [0, T]$ and for all $G \in C^{1,1}([0, T] \times \mathbb{T} \setminus \{b\})$*

$$\lim_{n \rightarrow +\infty} \int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s \rangle ds = \int_0^t \langle \partial_u \rho_s^*, G_s \rangle ds.$$

Proof. First we consider $G \in C^{0,1}([0, T] \times \mathbb{T})$ compactly support in $[0, T] \times (\mathbb{T} \setminus \{b\})$ such that $G_s(u) = 0$ for all $s \in [t, T]$ and all $u \in \mathbb{T}$. In this case,

$$\int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s \rangle ds = - \int_0^t \langle \rho_s^{\alpha_n}, \partial_u G_s \rangle ds.$$

By the previous equality and since ρ^{α_n} converges to ρ^* in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$, as $n \rightarrow +\infty$, we obtain that

$$\lim_{n \rightarrow +\infty} \int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s \rangle ds = \int_0^t \langle \partial_u \rho_s^*, G_s \rangle ds.$$

The next step is to extend the previous equality to functions $G \in C^{1,1}([0, T] \times \mathbb{T} \setminus \{b\})$. For that purpose, fix $G \in C^{1,1}([0, T] \times \mathbb{T} \setminus \{b\})$ and fix $t \in [0, T]$. Approximate G in $L^2([0, T] \times \mathbb{T})$ by a function $G^\varepsilon \in C^{1,1}([0, T] \times \mathbb{T})$ with compact support in $[0, T] \times (\mathbb{T} \setminus \{b\})$ and such that $\|G^\varepsilon\|_\infty \leq \|G\|_\infty$. For $\delta > 0$, let us define the function $\varphi^\delta : [0, T] \rightarrow \mathbb{R}$ as

$$\varphi^\delta(s) = \begin{cases} 1, & \text{if } s \in [0, t - \delta], \\ \frac{t-s}{\delta}, & \text{if } s \in [t - \delta, t], \\ 0, & \text{if } s \in [t, T]. \end{cases}$$

Denote $G_s^{\varepsilon, \delta}(u) := G_s^\varepsilon(u) \varphi^\delta(s)$. Then, $G^{\varepsilon, \delta} \in C^{0,1}([0, T] \times \mathbb{T})$ has compact support in $[0, T] \times (\mathbb{T} \setminus \{b\})$ and $G_s^{\varepsilon, \delta}(u) = 0$ for all $s \in [t, T]$ and for all $u \in \mathbb{T}$. Therefore,

$$\lim_{n \rightarrow \infty} \int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s^{\varepsilon, \delta} \rangle ds = \int_0^t \langle \partial_u \rho_s^*, G_s^{\varepsilon, \delta} \rangle ds. \quad (26)$$

By the triangular inequality, we have that

$$\begin{aligned} & \left| \int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s \rangle ds - \int_0^t \langle \partial_u \rho_s^*, G_s \rangle ds \right| \leq \left| \int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s - G_s^\varepsilon \rangle ds \right| \\ & + \left| \int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s^\varepsilon - G_s^{\varepsilon, \delta} \rangle ds \right| + \left| \int_0^t \langle \partial_u \rho_s^{\alpha_n} - \partial_u \rho_s^*, G_s^{\varepsilon, \delta} \rangle ds \right| \\ & + \left| \int_0^t \langle \partial_u \rho_s^*, G_s^{\varepsilon, \delta} - G_s^\varepsilon \rangle ds \right| + \left| \int_0^t \langle \partial_u \rho_s^*, G_s^\varepsilon - G_s \rangle ds \right|. \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\left| \int_0^t \langle \partial_u \rho_s^{\alpha_n}, (G_s - G_s^\varepsilon) \rangle ds \right| \leq \left(\int_0^t \|\partial_u \rho_s^{\alpha_n}\|_{L^2(\mathbb{T})}^2 ds \right)^{1/2} \left(\int_0^t \|G_s - G_s^\varepsilon\|_{L^2(\mathbb{T})}^2 ds \right)^{1/2}.$$

By Theorem 2.2, ρ^{α_n} satisfies

$$\int_0^T \langle \rho_s^{\alpha_n}, \partial_u G_s \rangle ds - 2 \int_0^T \langle G_s, G_s \rangle ds \leq K_0,$$

for all $G \in C_c(b)$. This, together with Corollary A.5, assures that

$$\int_0^t \|\partial_u \rho_s^{\alpha_n}\|_{L^2(\mathbb{T})}^2 ds \leq 8K_0.$$

Thus,

$$\left| \int_0^t \langle \partial_u \rho_s^{\alpha_n}, (G_s - G_s^\varepsilon) \rangle ds \right| \leq (8K_0)^{1/2} \left(\int_0^t \|G_s - G_s^\varepsilon\|_{L^2(\mathbb{T})}^2 ds \right)^{1/2}.$$

By Proposition 6.2, the same holds for ρ^* , i.e., $\int_0^t \|\partial_u \rho_s^*\|_{L^2(\mathbb{T})}^2 ds \leq 8K_0$, hence

$$\left| \int_0^t \langle \partial_u \rho_s^*, (G_s - G_s^\varepsilon) \rangle ds \right| \leq (8K_0)^{1/2} \left(\int_0^t \|G_s - G_s^\varepsilon\|_{L^2(\mathbb{T})}^2 ds \right)^{1/2}.$$

Observe that

$$\begin{aligned} \left| \int_0^t \langle \partial_u \rho_s^{\alpha_n}, (G_s^{\varepsilon, \delta} - G_s^\varepsilon) \rangle ds \right| &= \left| \int_0^t \int_{\mathbb{T}} \partial_u \rho_s^{\alpha_n}(u) [G_s^\varepsilon(u) \varphi^\delta(s) - G_s^\varepsilon(u)] du ds \right| \\ &= \left| \int_{t-\delta}^t \left(\frac{t-s}{\delta} - 1 \right) \int_{\mathbb{T}} \partial_u \rho_s^{\alpha_n}(u) G_s^\varepsilon(u) du ds \right| \\ &\leq \int_0^T \mathbf{1}_{[t-\delta, t]}(s) \left| \int_{\mathbb{T}} \partial_u \rho_s^{\alpha_n}(u) G_s^\varepsilon(u) du \right| ds \\ &\leq \left(\int_0^T \mathbf{1}_{[t-\delta, t]}^2(s) ds \right)^{1/2} \left(\int_0^T \left| \int_{\mathbb{T}} \partial_u \rho_s^{\alpha_n}(u) G_s^\varepsilon(u) du \right|^2 ds \right)^{1/2} \\ &= \sqrt{\delta} \left(\int_0^T \left| \int_{\mathbb{T}} \partial_u \rho_s^{\alpha_n}(u) G_s^\varepsilon(u) du \right|^2 ds \right)^{1/2} \\ &\leq \sqrt{\delta} \|G^\varepsilon\|_\infty \left(\int_0^T \left| \int_{\mathbb{T}} \partial_u \rho_s^{\alpha_n}(u) du \right|^2 ds \right)^{1/2} \\ &\leq \sqrt{\delta} \|G^\varepsilon\|_\infty \left(\int_0^T \int_{\mathbb{T}} (\partial_u \rho_s^{\alpha_n}(u))^2 du ds \right)^{1/2} \\ &\leq \sqrt{\delta} \|G\|_\infty (8K_0)^{1/2}. \end{aligned}$$

By analogous calculations, we get that

$$\left| \int_0^t \langle \partial_u \rho_s^*, (G_s^{\varepsilon, \delta} - G_s^\varepsilon) \rangle ds \right| \leq \sqrt{\delta} \|G\|_\infty (8K_0)^{1/2}. \quad (27)$$

Putting together the previous bounds, we get

$$\begin{aligned} \left| \int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s \rangle ds - \int_0^t \langle \partial_u \rho_s^*, G_s \rangle ds \right| &\leq \left| \int_0^t \langle \partial_u \rho_s^{\alpha_n}, G_s^{\varepsilon, \delta} \rangle ds - \int_0^t \langle \partial_u \rho_s^*, G_s^{\varepsilon, \delta} \rangle ds \right| \\ &\quad + 2(8K_0)^{1/2} \left\{ \left(\int_0^t \|G_s - G_s^\varepsilon\|_{L^2(\mathbb{T})}^2 ds \right)^{1/2} + \sqrt{\delta} \|G\|_\infty \right\}. \end{aligned}$$

Now, by (26) the proof follows. \square

Proposition 6.6. *Let $\{\alpha_n : n \geq 1\}$ be a sequence of positive numbers such that $\lim_{n \rightarrow +\infty} \alpha_n = 0$. If $\{\rho^{\alpha_n} : n \geq 1\}$ converges to ρ^* in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$, as $n \rightarrow +\infty$, then ρ^* is the unique weak solution of (10).*

Proof. We prove that ρ^* satisfies the two conditions of Definition 10. From Proposition 6.2 we have that ρ^* satisfies condition (1). In order to prove condition (2), the main idea is to take the limit as $n \rightarrow +\infty$ in (25) and to analyze the limiting terms.

A simple computation shows that for $t \in [0, T]$:

$$\left| \int_0^t \alpha_n \{ \rho_s^{\alpha_n}(b^+) - \rho_s^{\alpha_n}(b^-) \} \{ H_s(b^+) - H_s(b^-) \} ds \right| \leq 4T \|H\|_\infty \alpha_n,$$

therefore, when $\alpha_n \rightarrow 0$, last integral in (25) converges to zero, as $n \rightarrow +\infty$. Now, replacing ρ^α by ρ^{α_n} in (25), recalling the first statement of Proposition 6.4 and Proposition 6.5, and sending $n \rightarrow +\infty$, we conclude that ρ^* satisfies:

$$\langle \rho_t^*, H_t \rangle - \langle \rho_0, H_0 \rangle + \int_0^t \langle \partial_u \rho_s^*, \partial_u H_s \rangle ds - \int_0^t \langle \rho_s^*, \partial_s H_s \rangle ds = 0,$$

for $t \in [0, T]$ almost surely and for $H \in \mathcal{A}$, where \mathcal{A} is a countable dense subset of $C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$.

Since $\rho^* \in L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$ and performing integration by parts in the previous equation, we get to:

$$\begin{aligned} & \langle \rho_t^*, H_t \rangle - \langle \rho_0, H_0 \rangle - \int_0^t \langle \rho_s^*, \Delta H_s + \partial_s H_s \rangle ds \\ & - \int_0^t \{ \rho_s^*(b^+) \partial_u H_s(b^+) - \rho_s^*(b^-) \partial_u H_s(b^-) \} ds = 0. \end{aligned}$$

for $t \in [0, T]$ almost surely and for $H \in \mathcal{A}$, where \mathcal{A} is a countable dense subset of $C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$. According to the second statement of Proposition 6.4, the term $\langle \rho_t^*, H_t \rangle$ is a continuous function in $t \in [0, T]$. The remaining parcels in the equation above are clearly continuous functions in $t \in [0, T]$. Hence, given $H \in \mathcal{A}$, the equality holds for all $t \in [0, T]$ and since \mathcal{A} is dense, the equation above holds for $H \in C^{1,2}([0, T] \times \overline{\mathbb{T} \setminus \{b\}})$. \square

Proposition 6.7. *Let $\{\alpha_n : n \geq 1\}$ be a sequence of positive numbers such that $\lim_{n \rightarrow +\infty} \alpha_n = +\infty$. If $\{\rho^{\alpha_n} : n \geq 1\}$ converges to ρ^* in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$, as $n \rightarrow +\infty$, then ρ^* is the unique weak solution of (7).*

Proof. In order to prove that ρ^* satisfies the integral equation (11), as above, the main idea is to take the limit as $n \rightarrow +\infty$ in (25) and to analyze the limiting terms. In this situation, we take $H \in C^{1,2}([0, T] \times \mathbb{T})$, so that (25) is given by

$$\langle \rho_t^{\alpha_n}, H_t \rangle - \langle \rho_0, H_0 \rangle + \int_0^t \langle \partial_u \rho_s^{\alpha_n}, \partial_u H_s \rangle ds - \int_0^t \langle \rho_s^{\alpha_n}, \partial_s H_s \rangle ds = 0.$$

By the first statement of Proposition 6.4 and Proposition 6.5 and sending $n \rightarrow +\infty$ in the previous equality, we conclude that ρ^* satisfies:

$$\langle \rho_t^*, H_t \rangle - \langle \rho_0, H_0 \rangle + \int_0^t \langle \partial_u \rho_s^*, \partial_u H_s \rangle ds - \int_0^t \langle \rho_s^*, \partial_s H_s \rangle ds = 0, \quad (28)$$

for $t \in [0, T]$ almost surely and $H \in \mathcal{A}$, where \mathcal{A} is a dense subset of $C^{1,2}([0, T] \times \mathbb{T})$.

According to the second statement of Proposition 6.4, $\langle \rho_t^*, H_t \rangle$ is a continuous function in $t \in [0, T]$. Since, the remaining terms in (28) are also continuous functions in $t \in [0, T]$, then (28) holds, in fact, for all $t \in [0, T]$. By density arguments, (28) holds for all $H \in C^{1,2}([0, T] \times \mathbb{T})$.

Now, to obtain equation (11) from (28), we must to handle the integral term $\int_0^t \langle \partial_u \rho_s^*, \partial_u H_s \rangle ds$. By Proposition 6.5 and then performing integration by parts, we have that

$$\begin{aligned} \int_0^t \langle \partial_u \rho_s^*, \partial_u H_s \rangle ds &= \lim_{\alpha_n \rightarrow +\infty} \int_0^t \langle \partial_u \rho_s^{\alpha_n}, \partial_u H_s \rangle ds \\ &= \lim_{\alpha_n \rightarrow +\infty} \left\{ \int_0^t \langle \rho_s^{\alpha_n}, \Delta H_s \rangle ds - \int_0^t \{ \rho_s^{\alpha_n}(b^+) - \rho_s^{\alpha_n}(b^-) \} \partial_u H_s(b) ds \right\}. \end{aligned}$$

We claim that the previous limit is equal to $\int_0^t \langle \rho_s^*, \Delta H_s \rangle ds$. At first we prove that

$$\lim_{\alpha_n \rightarrow +\infty} \int_0^t \{ \rho_s^{\alpha_n}(b^+) - \rho_s^{\alpha_n}(b^-) \} \partial_u H_s(b) ds = 0.$$

By Cauchy-Schwarz inequality and Proposition A.6,

$$\int_0^t \{ \rho_s^\alpha(b^+) - \rho_s^\alpha(b^-) \} \partial_u H_s(b) ds \leq \left(\int_0^T (\partial_u H_s(b))^2 ds \right)^{1/2} \frac{1}{\alpha} \left(\int_0^T (\partial_u \rho_s^\alpha(b))^2 ds \right)^{1/2},$$

for all $t \in [0, T]$. Without loss of generality, we can assume $\alpha \geq 1$. Thus, by Theorem 2.2,

$$\langle \rho^\alpha, \partial_u H \rangle - 2 \langle H, H \rangle_{W_1} \leq K_0,$$

for all $H \in C^{0,1}([0, T] \times \mathbb{T})$. From Proposition A.4,

$$\sup_{H \in C^{0,1}([0, T] \times \mathbb{T})} \left\{ \langle \rho^\alpha, \partial_u H \rangle - 2 \langle H, H \rangle_{W_1} \right\} = \frac{1}{8} \int_0^T \left\{ \|\partial_u \rho_s^\alpha\|_{L^2(\mathbb{T})}^2 + (\partial_u \rho_s^\alpha(b))^2 \right\} ds.$$

Therefore,

$$\int_0^t \{ \rho_s^\alpha(b^+) - \rho_s^\alpha(b^-) \} \partial_u H_s(b) ds \leq \frac{1}{\alpha} (8K_0)^{1/2} \left(\int_0^T (\partial_u H_s(b))^2 ds \right)^{1/2},$$

for all $t \in [0, T]$ and $\alpha \geq 1$.

In order to finish the proof is is enough to show that

$$\lim_{\alpha_n \rightarrow +\infty} \int_0^t \langle \rho_s^{\alpha_n}, \Delta H_s \rangle ds = \int_0^t \langle \rho_s^*, \Delta H_s \rangle ds.$$

This is an easy consequence of Cauchy-Schwarz inequality together with the fact that $\rho_t^{\alpha_n}$ converges to ρ_t^* in $L^2(\mathbb{T})$. □

Proof of Theorem 2.3. As mentioned in the beginning of Subsection 6, the set $\{\rho^\alpha : \alpha > 0\}$ is compactly embedded in $L^2(0, T; L^2(\mathbb{T} \setminus \{b\}))$. Therefore, any sequence $\alpha_n \rightarrow 0$ has a subsequence α_{n_j} such that $\rho^{\alpha_{n_j}}$ converges to some ρ^* . By Proposition 6.2, Proposition 6.6 and from uniqueness of weak solutions of (10), we conclude that ρ^* is the unique weak solution of (10). Hence, $\lim_{\alpha \rightarrow 0} \rho^\alpha = \rho^*$. Using Proposition 6.7 and analogous arguments, we get that $\lim_{\alpha \rightarrow \infty} \rho^\alpha = \hat{\rho}$, where $\hat{\rho}$ is the unique weak solution of (7). □

APPENDIX A. SOBOLEV SPACE TOOLS

For completeness, in this section we prove some results that we use along the paper.

Proposition A.1. *The set $C^{0,0}([0, T] \times \mathbb{T})$ is a dense subset of $L^2_{W_\alpha}([0, T] \times \mathbb{T})$.*

Proof. Let $H \in L^2_{W_\alpha}([0, T] \times \mathbb{T})$. Then, $H \in L^2([0, T] \times \mathbb{T})$ and $H(\cdot, b) \in L^2([0, T])$. Consider a sequence $\{H_n : n \geq 1\}$, such that for each $n \in \mathbb{N}$, $H_n \in C^{0,0}([0, T] \times \mathbb{T})$ with compact support in $[0, T] \times (\mathbb{T} \setminus \{b\})$ converging in $L^2([0, T] \times \mathbb{T})$ to H , as $n \rightarrow +\infty$. Consider also a sequence $\{h_n : n \geq 1\}$, of continuous functions $h_n : [0, T] \rightarrow \mathbb{R}$ and converging in $L^2([0, T])$ to $H(\cdot, b)$, as $n \rightarrow +\infty$. Let

$$G_n(t, u) := H_n(t, u) + h_n(t) \mathbf{1}_{(b-\frac{1}{n}, b+\frac{1}{n})}(u).$$

Noticing that for each $n \geq 1$

$$\|G_n - H\|_{W_\alpha}^2 = \|H_n - H\|_{L^2(\mathbb{T})}^2 + \frac{2}{\alpha n} \int_0^T (H(t, b) - h_n(t))^2 dt$$

the proof ends. \square

Proposition A.2. *Let $\xi : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ be such that*

$$\sup_{H \in C^{0,1}([0, T] \times \mathbb{T})} \langle \partial_u H, \xi \rangle - \kappa \langle H, H \rangle_{W_\alpha} < \infty,$$

for some $\kappa > 0$. Then, there exists a function in $L^2_{W_\alpha}([0, T] \times \mathbb{T})$, which we denote by $\partial_u \xi$, such that

$$\langle \partial_u H, \xi \rangle = -\langle H, \partial_u \xi \rangle_{W_\alpha} = -\langle H, \partial_u \xi \rangle - \frac{1}{\alpha} \int_0^T H_t(b) \partial_u \xi_t(b) dt, \quad (29)$$

for all $H \in C^{0,1}([0, T] \times \mathbb{T})$.

Proof. Following the same arguments as in the proof of Proposition 6.2, this is a consequence of Riesz Representation Theorem. \square

Remark A.3. *The function $\partial_u \xi$ above is indeed the weak derivative of the function ξ in the usual sense. To see this, note that, for $H \in C_c(b)$,*

$$\langle \partial_u H, \xi \rangle = -\langle H, \partial_u \xi \rangle_{W_\alpha} = -\langle H, \partial_u \xi \rangle.$$

Proposition A.4. *Let $\xi : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ be such that*

$$\sup_{H \in C^{0,1}([0, T] \times \mathbb{T})} \left\{ \langle \partial_u H, \xi \rangle - \kappa \langle H, H \rangle_{W_\alpha} \right\} < \infty,$$

for some $\kappa > 0$. Then,

$$\begin{aligned} \sup_{H \in C^{0,1}([0, T] \times \mathbb{T})} \left\{ \langle \partial_u H, \xi \rangle - \kappa \langle H, H \rangle_{W_\alpha} \right\} &= \frac{1}{4\kappa} \int_0^T \|\partial \xi_t\|_{L^2_{W_\alpha}}^2 dt \\ &= \frac{1}{4\kappa} \int_0^T \left\{ \|\partial \xi_t\|_{L^2(\mathbb{T})}^2 + \frac{1}{\alpha} (\partial_u \xi_t(b))^2 \right\} dt. \end{aligned} \quad (30)$$

Proof. By Proposition A.2, for all $H \in C^{0,1}([0, T] \times \mathbb{T})$,

$$\langle \partial_u H, \xi \rangle - \kappa \langle H, H \rangle_{W_\alpha} = -\langle H, \partial_u \xi \rangle_{W_\alpha} - \kappa \langle H, H \rangle_{W_\alpha}. \quad (31)$$

By Young's Inequality, for all $r > 0$,

$$|\langle\langle H, \partial_u \xi \rangle\rangle_{W_\alpha}| \leq \frac{r}{2} \langle\langle H, H \rangle\rangle_{W_\alpha} + \frac{1}{2r} \langle\langle \partial_u \xi, \partial_u \xi \rangle\rangle_{W_\alpha}.$$

Choosing $r = 2\kappa$, together with (31) we get to

$$\sup_{H \in C^{0,1}([0,T] \times \mathbb{T})} \left\{ \langle\langle \partial_u H, \xi \rangle\rangle - \kappa \langle\langle H, H \rangle\rangle_{W_\alpha} \right\} \leq \frac{1}{4\kappa} \int_0^T \|\partial_u \xi_t\|_{L_{W_\alpha}^2}^2 dt.$$

For the reversed inequality, let $\{H^n : n \geq 1\} \subset C^{0,1}([0,T] \times \mathbb{T})$ converging to $r \partial_u \xi$ in $L_{W_\alpha}^2([0,T] \times \mathbb{T})$, as $n \rightarrow +\infty$. The constant $r \in \mathbb{R}$ will be chosen ahead. Thus,

$$\begin{aligned} & \sup_{H \in C^{0,1}([0,T] \times \mathbb{T})} \left\{ \langle\langle \partial_u H, \xi \rangle\rangle - \kappa \langle\langle H, H \rangle\rangle_{W_\alpha} \right\} \\ &= \sup_{H \in C^{0,1}([0,T] \times \mathbb{T})} \left\{ -\langle\langle H, \partial_u \xi \rangle\rangle_{W_\alpha} - \kappa \langle\langle H, H \rangle\rangle_{W_\alpha} \right\} \\ &\geq \lim_{n \rightarrow +\infty} \left\{ -\langle\langle H^n, \partial_u \xi \rangle\rangle_{W_\alpha} - \kappa \langle\langle H^n, H^n \rangle\rangle_{W_\alpha} \right\} \\ &= (-r - \kappa r^2) \int_0^T \|\partial_u \xi_t\|_{L_{W_\alpha}^2}^2 dt. \end{aligned}$$

Taking $r = -\frac{1}{2\kappa}$ the proof ends. \square

Corollary A.5. *Let $\xi \in L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$ be such that*

$$\sup_{H \in C_c(b)} \left\{ \langle\langle \partial_u H, \xi \rangle\rangle - \kappa \langle\langle H, H \rangle\rangle \right\} < \infty,$$

for some $\kappa > 0$. Then,

$$\sup_{H \in C_c(b)} \left\{ \langle\langle \partial_u H, \xi \rangle\rangle - \kappa \langle\langle H, H \rangle\rangle \right\} = \frac{1}{4\kappa} \int_0^T \|\partial_u \xi_t\|_{L^2(\mathbb{T})}^2 dt,$$

where the function $\partial_u \xi_t$ coincides, Lebesgue almost surely, with the function $\partial_u \xi_t$ of Proposition A.4.

Proof. Since $\xi \in L^2(0, T; \mathcal{H}^1(\mathbb{T} \setminus \{b\}))$ and by Remark A.3, the result a consequence of Proposition A.4. \square

Proposition A.6. *Let $\xi : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ be such that*

$$\sup_{H \in C^{0,1}([0,T] \times \mathbb{T})} \langle\langle \partial_u H, \xi \rangle\rangle - \kappa \langle\langle H, H \rangle\rangle_{W_\alpha} < \infty,$$

for some $\kappa > 0$. Then, $t \in [0, T]$ almost surely,

$$\xi_t(v) - \xi_t(u) = \int_{[u,v)} \partial_u \xi_t(z) W_\alpha(dz), \quad \forall u, v \in \mathbb{T},$$

where $\partial_u \xi$ satisfies (29). In particular,

$$\xi_t(b^+) - \xi_t(b^-) = \frac{1}{\alpha} \partial_u \xi_t(b),$$

$t \in [0, T]$ almost surely.

Proof. From Proposition A.2, for $t \in [0, T]$ almost surely,

$$\langle\langle \partial_u H, \xi_t \rangle\rangle = -\langle\langle H, \partial_u \xi_t \rangle\rangle_{W_\alpha}, \quad (32)$$

for all $H \in C^1(\mathbb{T})$. For $u, v \in \mathbb{T}$ and $n \geq 1$ define $f_n : \mathbb{T} \rightarrow \mathbb{R}$ by

$$f_n(w) = \begin{cases} -W_\alpha([u, u + \frac{1}{n}])^{-1}, & \text{if } w \in [u, u + \frac{1}{n}] \\ W_\alpha([v, v + \frac{1}{n}])^{-1}, & \text{if } w \in [v, v + \frac{1}{n}] \\ 0, & \text{otherwise,} \end{cases}$$

and $H_n : \mathbb{T} \rightarrow \mathbb{R}$ by

$$H_n(z) = \int_{(0, z]} f_n(w) dW_\alpha(w).$$

Since for each $n \geq 1$, f_n is not a continuous function, then $H_n \notin C^1(\mathbb{T})$. However, by approximating for each $n \geq 1$, f_n and H_n by continuous functions $f_n^\varepsilon : \mathbb{T} \rightarrow \mathbb{R}$ and $H_n^\varepsilon(r) = \int_{(0, r]} f_n^\varepsilon(z) W_\alpha(dz)$, respectively, equality (32) is still valid for f_n and H_n .

We claim that $H_n(r)$ converges to $-\mathbf{1}_{[u, v)}(r)$, W_α -almost surely, as $n \rightarrow +\infty$. Indeed, if $u, v \neq b$, $H_n(r)$ converges pointwise to $-\mathbf{1}_{(u, v)}(r)$, which is equal to $-\mathbf{1}_{[u, v)}(r)$, W_α -almost surely, as $n \rightarrow +\infty$. If $u = b$, then $H_n(r)$ converges pointwise to $-\mathbf{1}_{[u, v)}(r)$, which is equal to $-\mathbf{1}_{(u, v)}(r)$, W_α -almost surely, as $n \rightarrow +\infty$. If $v = b$, then $H_n(r)$ converges pointwise to $-\mathbf{1}_{(u, v)}(r)$, which is equal to $-\mathbf{1}_{[u, v)}(r)$, W_α -almost surely, as $n \rightarrow +\infty$. By Cauchy-Schwarz inequality,

$$\lim_{n \rightarrow +\infty} \langle -\partial_u \xi_t, H_n \rangle_{W_\alpha} = \langle \partial_u \xi_t, \mathbf{1}_{[u, v)} \rangle_{W_\alpha}.$$

By the definition of f_n , we have that for each $n \geq 1$

$$\begin{aligned} \langle \xi_t, f_n \rangle_{W_\alpha} &= \frac{1}{W_\alpha([v, v + \frac{1}{n}])} \int_{[v, v + \frac{1}{n}]} \xi_t(w) W_\alpha(dw) \\ &\quad - \frac{1}{W_\alpha([u, u + \frac{1}{n}])} \int_{[u, u + \frac{1}{n}]} \xi_t(w) W_\alpha(dw). \end{aligned}$$

Sending $n \rightarrow +\infty$ in the previous equality, by Lebesgue-Besicovitch Differentiation Theorem (see [2]) we obtain that $\langle \xi_t, f_n \rangle_{W_\alpha}$ converges to $\xi_t(v) - \xi_t(u)$, W_α -almost surely in $u, v \in \mathbb{T}$, as $n \rightarrow +\infty$, which finishes the proof of the first claim. Finally,

$$\xi_t(b^+) - \xi_t(b^-) = \lim_{\substack{u \rightarrow b^- \\ v \rightarrow b^+}} \int_{[u, v]} \partial_u \xi_t(w) W_\alpha(dw) = \frac{1}{\alpha} \partial_u \xi_t(b).$$

□

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